2 Fundamental PDEs

1.[Ex.1] Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1.1}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Proof. Consider $v(s) = u(bs + x_0, s)$, then

$$\frac{d}{ds} v(s) = u_t(bs + x_0, s) + b \cdot Du(bs + x_0, s) = -cu(bs + x_0, s) = -cv(s)$$
$$v(0) = g(x_0)$$

It is an initial value problem of ODE, and we could solve out v:

$$\frac{d}{ds}(e^{cs} v(s)) = 0 \implies v(t) = e^{-ct} g(x_0)$$

Thus, $u(bt + x_0, t) = v(t) = e^{-ct} g(x_0)$ for $\forall x_0 \in \mathbb{R}^n,\ t \in \mathbb{R}$. Let $x = x_0 + bt$, we get

$$u(x, t) = e^{-ct} g(x - bt) \tag{1.2}$$

□

2.[Ex.2] Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) = u(Ox) \quad (x \in \mathbb{R}^n), \tag{2.1}$$

then $\Delta v = 0$

Proof. Let $O = (o_{ij})_{1 \le i,j \le n}$. Then O is orthogonal $\iff \sum_{k=1}^n o_{ik} o_{jk} = \delta_{ij}$. Where $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$. Exploit Chain's rule:

$$\partial_i v(x) = \partial_i [u(Ox)] = \sum_{j=1}^n \partial_j u \cdot o_{ji}$$
$$\partial_i^2 v(x) = \partial_i \Big(\sum_{j=1}^n \partial_j u(Ox) \cdot o_{ji} \Big) = \sum_{j,k=1}^n \partial_{jk} u \cdot (o_{ji} o_{ki})$$
$$\Delta v = \sum_{i=1}^n \partial_i^2 v = \sum_{j,k=1}^n \partial_{jk} u \cdot (\sum_{i=1}^n o_{ji} o_{ki}) = \Delta u = 0$$

□

3.[Ex.3] Modify the proof of the mean-value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g\ dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} (\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}) f dx \tag{3.1}$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases} \tag{3.2}$$

Proof. Define $\psi(r) = \fint_{\partial B(0,r)} u\ dS$, then

$$u(0) = \lim_{r\to 0} \psi(r) = \psi(r) - \int_0^r \psi'(s)\ ds = \fint_{\partial B(0,r)} g\ dS - \int_0^r \psi'(s)\ ds$$

$$\psi'(r) = \fint_{\partial B(0,r)} \frac{\partial u}{\partial n} dS = \frac{1}{r^{n-1} n\alpha(n)} \int_{B(0,r)} \Delta u\ dx = \frac{-1}{r^{n-1} n\alpha(n)} \int_{B(0,r)} f(x)\ dx$$

$$\begin{aligned}
-\int_0^r \psi'(s)\ ds &= \frac{1}{n\alpha(n)} \int_0^r \frac{1}{s^{n-1}} \int_{B(0,s)} f(x) dx\ ds \\
&= \frac{1}{n(n-2)\alpha(n)} \int_0^r - \int_{B(0,s)} f dx\ ds^{2-n} \\
&= \frac{1}{n(n-2)\alpha(n)} \Big(- \frac{1}{r^{n-2}} \int_{B(0,r)} f dx + \lim_{s\to 0} \frac{1}{s^{n-2}} \int_{B(0,s)} f dx + \\
&\int_0^r \frac{1}{s^{2-n}} \frac{d}{ds} (\int_{B(0,s)} f dx) ds \Big) \\
&= \frac{1}{n(n-2)\alpha(n)} \Big(- \frac{1}{r^{n-2}} \int_{B(0,r)} f dx + \int_0^r \frac{1}{s^{n-2}} \int_{\partial B(0,s)} f\ dS ds \Big) \\
&= \frac{1}{n(n-2)\alpha(n)} \Big(- \frac{1}{r^{n-2}} \int_{B(0,r)} f dx + \int_0^r \int_{\partial B(0,s)} \frac{1}{|x|^{n-2}} f\ dS ds \Big) \\
&= \frac{1}{n(n-2)\alpha(n)} \Big(- \frac{1}{r^{n-2}} \int_{B(0,r)} f dx + \int_{B(0,r)} \frac{1}{|x|^{n-2}} f\ dx \Big) \\
&= \frac{1}{n(n-2)\alpha(n)} \int_{B(x,r)} (\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}) f\ dx
\end{aligned}$$

Hence,

$$u(0) = \fint_{\partial B(0,r)} g\ dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} (\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}) f dx$$

□

4.[Ex.4] Give a direct proof that if $u \in C^2(U) \cap C(\bar{U})$ is harmonic within a bounded open set U, then

$$\max_{\bar{U}} u = \max_{\partial U} u$$

(Hint: Define $u_\varepsilon := u + \varepsilon|x|^2$ for $\varepsilon > 0$, and show u_ε cannot attain its maximum over $\bar{U}$ at an interior point.)

Proof. Let u_ε defined as above., Then

$$\Delta u_\varepsilon = \Delta u + n\varepsilon > 0$$

Assume u_ε attains its maximum at an interior point x_0. Then $\partial_i u_\varepsilon(x_0) = 0$, $\partial_i^2 u_\varepsilon(x_0) \leq 0$. Hence, $\Delta u_\varepsilon(x_0) \leq 0$, which contradicts to $\Delta u_\varepsilon > 0$. Thus, we have

$$\max_{\partial U} u \leq \max_{\bar{U}} u \leq \max_{\bar{U}} u_\varepsilon = \max_{\partial U} u_\varepsilon \leq \max_{\partial U} u + \varepsilon \max_{\bar{U}} |x|^2$$

Let $\varepsilon \to 0$, we get $\max_{\partial U} u \leq \max_{\bar{U}} u \leq \max_{\partial U} u$, which implies they are identical. □

5.[Ex.5] We say $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in } U$$

(a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v \, dy \quad \text{for all } B(x,r) \subset U \tag{5.1}$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Proof. For (a): define $\psi(r) = \fint_{\partial B(x,r)} v \, dy$. Then by the same calculation of Ex4. and proof of mean-value thm, we have

$$\psi'(r) = C \int_{B(x,r)} \Delta v \, dy \geq 0, \qquad \frac{n}{r^n} \int_0^r s^{n-1} \, ds = 1$$

$$v(x) = \lim_{r \to 0} \psi(r) \leq \frac{n}{r^n} \int_0^r s^{n-1} \psi(s) \, ds = \fint_{B(x,r)} v \, dy$$

For (b): Follows the same way as Ex.4

For (c): ϕ is convex, thus $\phi'' \geq 0$

$$\Delta v = \phi''(u)|Du|^2 + \phi'(u)\Delta u \geq 0$$

For (d):

$$\Delta v = 2 \sum_{1\leq i,j\leq n} (\partial_{ij}u)^2 + \partial_i u \partial_{ijj} u = 2 \sum_{1\leq i,j\leq n} (\partial_{ij}u)^2 \geq 0$$

□

6.[Ex.6] Let U be a bounded, open subset of $\mathbb{R}^n$, Prove that there exists a constant C, depending only on U, such that

$$\max_{\bar{U}} |u| \leq C(\max_{\partial U} |g| + \max_{\bar{U}} f)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases} \tag{6.1}$$

(Hint: $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leq 0$, for $\lambda := \max_{\bar{U}} f$)

Proof. Let $v := u + \frac{|x|^2}{2n}\lambda$, then $\Delta v = \Delta u + \lambda \geq 0$, thus v is subharmonic. Apply the maximum principle to v. Assume $M = \max_{\bar{U}} |x|^2$, we get

$$\max_{\bar{U}} |u| \leq \max_{\bar{U}} |v| + \frac{M}{2n}\lambda = \max_{\partial U} |v| + \frac{M}{2n}\lambda \leq \max_{\partial U} |g| + \frac{M}{n}\lambda \tag{6.2}$$

Let $C = \max\{1, \frac{M}{n}\}$, we conclude. □

7.[Ex.7] Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r - |x|}{(r + |x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r + |x|}{(r - |x|)^{n-1}}u(0) \tag{7.1}$$

whenever u is positive and harmonic in $B^0(0, r)$. This is an explicit form of Harnack's inequality.

Proof. Recall the Poisson's formula for the ball:

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x - y|^n} \, dS(y) \quad \forall x \in B^0(0, r) \tag{7.2}$$

where $u = g \geq 0$ on $\partial B(0,r)$. By mean-value theorem, we have

$$u(0) = \fint_{\partial B(0,r)} g(y)\ dS(y) \tag{7.3}$$

By Poisson's formula (7.2):

$$\begin{aligned} u(x) &\leq \frac{r^2-|x|^2}{n\alpha(n)r(r-|x|)^n}\int_{\partial B(0,r)} g(y)\ dS(y) = r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0) \\ u(x) &\geq \frac{r^2-|x|^2}{n\alpha(n)r(r+|x|)^n}\int_{\partial B(0,r)} g(y)\ dS(y) = r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \end{aligned}$$

□

8.[Ex.8] Prove Theorem 15 in §2.2.4. (Hint: SInce $u \equiv 1$ solves (44) for $g \equiv 1$, the theory automatically implies

$$\int_{\partial B(0,1)} K(x,y)\ dS(y) = 1 \tag{8.1}$$

for each $x \in B^0(0,1)$)

Proof. For (i): by Poisson's formula:

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\ dS(y) \quad \forall x \in B^0(0,r)$$

Since $1/|x-y|^n$ is C^∞ and $\int_{\partial B(0,r)} |D^k(1/|x-y|^n)|\ dS(y) < \infty$, by Fubini's theorem, we know u is C^∞.
For (ii): since u is C^∞, we have

$$\Delta u = \frac{1}{n\alpha(n)r}\int_{\partial B(0,r)} \Delta_x(\frac{r^2-|x|^2}{|x-y|^n})g(y)\ dS(y)$$

By direct calculation we have

$$\Delta_x(\frac{r^2-|x|^2}{|x-y|^n}) = \frac{-2n}{|x-y|^n} + \frac{4nx\cdot(x-y)}{|x-y|^{n+2}} + \frac{2n(r^2-|x|^2)}{|x-y|^{n+2}} = 0$$

For (iii): since g is continuous, suppose $\varepsilon > 0$, there exists $\delta > 0$, such that $|g(x)-g(y)| < \varepsilon$ when $\|x-y\| \leq \delta$. Then by (8.1), we have

$$\begin{aligned} |u(x)-g(x_0)| &= \frac{r^2-|x|^2}{n\alpha(n)r}|\int_{\partial B(0,r)} \frac{g(y)-g(x_0)}{|x-y|^n}\ dS(y)| \\ &\leq \frac{r-|x|^2}{n\alpha(n)r}(\int_{|y-x|<\delta} \frac{|g(y)-g(x_0)|}{|x-y|^n} dS(y) + \int_{|y-x|\geq\delta} \frac{2\sup|g|}{\delta^n} dS(y) \end{aligned}$$

$$\leq \varepsilon + \frac{2|\partial B(0,r)| \cdot \sup |g|}{n\alpha(n)r}(r^2 - |x|^2)$$

Thus when $x \to x_0$, $|x| \to r$. We have

$$\lim_{x \to x_0} |u(x) - g(x_0)| \leq \varepsilon \quad \text{for } \forall \varepsilon > 0$$

□

9.[Ex.9] Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{9.1}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \leq 1$. Show Du is *not* bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda e_n) - u(0)}{\lambda}$.)

Proof. Recall Poisson's formula for half-space:

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy$$

with $g(y) = |y| \in C(\mathbb{R}^{n-1})$, we have $u \in C^\infty(\mathbb{R}^n_+)$. Thus

$$\begin{aligned} \fint_0^\lambda \partial_n u(te_n)\, dt = \frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{|y|}{(\lambda + |y|)^n} dy \\ &= \frac{2|\partial B^{n-1}(0,1)|}{n\alpha(n)} \int_0^\infty \frac{r^{n-1}}{(\lambda + r)^n} dr \to \infty \quad as\ \lambda \to 0 \end{aligned}$$

Thus $\partial_n u$ can not be bounded near $x = 0$. □

10.[Ex.10] (Reflection principle)

(a) Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n \mid |x| < 1,\ x_n > 0\}$. Assume $u \in C^2(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases} \tag{10.1}$$

for $x \in U = B^0(0,1)$. Prove $v \in C^2(U)$ and thus v is harmonic within U.

(b) Now assume only that $u \in C^2(U^+) \cap C(\overline{U^+})$. Show that v is harmonic within U. (Hint: Use Poisson's formula for the ball.)

Proof. For (a), obviously v is C^2 in $\{x_n > 0\}$, $\{x_n < 0\}$ part, and $v \in C(U)$. As for $\{x_n = 0\}$ part:

$$\partial_i v(x) = \begin{cases} \partial_i u(x), & x_n \geq 0 \\ -\partial_i u(x', -x_n), & x_n < 0 \end{cases} \qquad i = 1, \ldots, n-1 \tag{10.2}$$

$$\partial_n v(x) = \begin{cases} \partial_n u(x), & x_n \geq 0 \\ \partial_n u(x', -x_n), & x_n < 0 \end{cases} \tag{10.3}$$

Since $u = 0$ at $\{x_n = 0\}$, $\partial_i u(x', 0) = 0$. Thus $\partial_i v$ is continuous at $\{x_n = 0\}$ for $i = 1, \ldots, n-1$. The continuity of $\partial_n v$ is trivial by (10.3). Hence, v is C^1. Keep going and notice that $\partial_{ij} u = 0$ at $\{x_n = 0\}$ for $i, j \neq n$. We could deduce that $D^2 v$, except for $\partial_n^2 v$, are continuous in U. Thanks to $\Delta u = 0$, we have

$$\partial_n^2 v(x) = \begin{cases} \partial_n^2 u(x) = \sum_{i=1}^{n-1} -\partial_i^2 u, & x_n \geq 0 \\ -\partial_n^2 u(x', -x_n) = \sum_{i=1}^{n-1} -\partial_i^2 u, & x_n < 0 \end{cases}$$

$\partial_i^2 u$ are continuous in U, thus $\partial_n^2 v$ is continuous. We deduce that $v \in C^2(U)$. Then v is harmonic by direct calculation:

$$\Delta v = \begin{cases} \Delta u(x) = 0, & x_n \geq 0 \\ -\Delta u(x', -x_n) = 0 & x_n < 0 \end{cases}$$

For (b): recall the Poisson's formula for a ball:

$$u(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x - y|^n} dS(y)$$

We define g as

$$g(y) = \begin{cases} u(y), & y_n \geq 0 \\ -u(y', -y_n), & y_n < 0 \end{cases} \tag{10.4}$$

$g \in C(\partial B(0,1))$ because $u \in C(\overline{U^+})$, $u(y', 0) = 0$. Let $\tilde{u}$ be solution given by Poisson's formula with boundary value g. Then $\tilde{u} = u$ on $\partial B(0,1) \cap \{y_n \geq 0\}$.

$$\tilde{u}(x', 0) = \frac{1 - |x'|^2}{n\alpha(n)} \left(\int_{\partial B \cap \{y_n \geq 0\}} \frac{g(y)}{|x' - y'|^n} dS(y) + \int_{\partial B \cap \{y_n < 0\}} \frac{-g(y', -y_n)}{|x' - y'|^n} dS(y) \right) = 0$$

Thus $\tilde{u} = u$ on ∂U^+, and they are both harmonic. By uniqueness of Laplace's equation for boundary value problems. We have $\tilde{u} = u$ in U^+. And by anti-symmetry of g with respect to x_n, we know $\tilde{u}(x', -x_n) = -\tilde{u}(x', x_n)$, which shows $\tilde{u} = v$. Then v is harmonic since $\tilde{u}$ is. □

11.[Ex.11] (Kelvin transform for Laplace's equation) The *Kelvin transform* $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\bar{u} := u(\bar{x})|\bar{x}|^{n-2} = u\Big(\frac{x}{|x|^2}\Big)|x|^{2-n} \qquad (x \neq 0) \tag{11.1}$$

where $\bar{x} = x/|x|^2$. Show that if u is harmonic, then so is $\bar{u}$.
(Hint: First show that $D_x\bar{x}(D_x\bar{x})^T = |\bar{x}|^4 I$. Then mapping $x \to \bar{x}$ is conformal, meaning angle preserving.)

Proof.

$$\partial_{x_i}\bar{x}_j = \frac{\delta_{ij}|x|^2 - 2x_i x_j}{|x|^4}$$

$$[(D_x\bar{x})(D_x\bar{x})^T]_{ij} = \sum_{k=1}^{n}(\partial_i\bar{x}_k \cdot \partial_j\bar{x}_k) = \delta_{ij}\frac{1}{|x|^4}$$

$$\Delta\bar{x}_j = \sum_i \frac{\delta_{ij}2x_i - 2x_j}{|x|^4} - 2\cdot\frac{\delta_{ij}x_i|x|^2 - 2x_i^2 x_j}{|x|^6} = \frac{(4-2n)x_j}{|x|^4}$$

Then mapping $x \to \bar{x}$ is conformal. Recall $\Delta(|x|^{2-n}) = 0$, then

$$\begin{aligned}
\Delta\bar{u}(x) &= \Delta(u(\bar{x}))|x|^{2-n} + 2\nabla(u(\bar{x}))\cdot\nabla(|x|^{2-n}) \\
&= |x|^{2-n}(\sum_{i,j,k}\partial_{ij}u\cdot\partial_k\bar{x}_i\partial_k\bar{x}_j + \sum_i \partial_i u\cdot\Delta\bar{x}_i) + \\
&\quad 2(2-n)(\sum_{i,j}\partial_i u\partial_j\bar{x}_i\cdot x_j)|x|^{-n} \\
&= \frac{1}{|x|^{n+2}}\Delta u + \sum_i \partial_i u\Big(|x|^{2-n}\Delta\bar{x}_i + 2(2-n)|x|^{-n}(-\frac{x_i}{|x|^2})\Big) \\
&= 0
\end{aligned}$$

□

12.[Ex.12] Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(a) Show $u_\lambda(x,t) \coloneqq u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show $v(x,t) \coloneqq x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

Proof. For (a):

$$(\partial_t - \Delta)u_\lambda(x,t) = \lambda^2 \partial_t u(\lambda x, \lambda^2 t) - \lambda^2 \Delta u(\lambda x, \lambda^2 t) = 0$$

For (b): Notice

$$v(x,t) = \frac{d}{d\lambda} u_\lambda(x,t)\Big|_{\lambda=1}$$

Since $(\lambda, x, t) \to u(\lambda, x, t)$ is smooth and solves the heat equation for every λ, then v solves the hear equation as well. ☐

13.[Ex.13] Assume $n = 1$ and $u(x,t) = v(\frac{x}{\sqrt{t}})$.

(a) Show

$$u_t = u_{xx}$$

if and only if

$$v'' + \frac{z}{2}v' = 0 \tag{13.1}$$

Show that the general solution of (13.1) is

$$v(z) = c\int_0^z e^{-s^2/4}\, ds + d \tag{13.2}$$

(b) Differentiate $u(x,t) = v(\frac{x}{\sqrt{t}})$ with respect to x and select the constant c properly, to obtain the fundamental solution Φ for $n = 1$. Explain why this procedure produces the fundamental solution. (Hint: What is the initial condition for u?)

Proof. For (a), we have

$$\begin{aligned} u_t - u_{xx} &= \partial_t v(\frac{x}{\sqrt{t}}) - \partial_x^2 v(\frac{x}{\sqrt{t}}) \\ &= -\frac{x}{2t^{3/2}}v'(\frac{x}{\sqrt{t}}) - \frac{1}{t}v''(\frac{x}{\sqrt{t}}) \\ &= -\frac{1}{t}\Big(v''(z) + \frac{z}{2}v'(z)\Big) \end{aligned}$$

where $z = x/\sqrt{t}$. Hence u solves wave equation if and only if v satisfies (13.1). To solve $v(z)$:

$$\begin{aligned} &\frac{d}{dz}(e^{z^2/4}v'(z)) = e^{z^2/4}(v'' + \frac{z}{2}v') = 0 \\ &\implies e^{z^2/4}v'(z) \equiv c \\ &\implies v(z) = d + \int_0^z v'(s)\, ds = c\int_0^z e^{s^2/4}\, ds + d \end{aligned}$$

with $c = v'(0)$, $d = v(0)$.

For (b):

$$\frac{d}{dx}u(x,t) = \frac{1}{\sqrt{t}}v'(\frac{x}{\sqrt{t}}) = \frac{c}{\sqrt{t}}\exp(-\frac{x^2}{4t})$$

Solves the heat equation. Let $c = 1/\sqrt{4\pi}$ to normalize the solution. Then it becomes the fundamental solution to heat equation. □

14.[Ex.14] Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on} \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{14.1}$$

where $c \in \mathbb{R}$

Proof. Consider the fundamental solution:

$$\Phi_c := \frac{1}{(4\pi t)^{n/2}} e^{\frac{|x|^2}{4t} - ct}$$

It is easy to verify that for $t > 0$, Φ_c is smooth and solves the equation

$$\partial_t \Phi_c - \Delta \Phi_c + c\Phi_c = 0$$

And $\Phi_c = e^{-ct}\Phi$, where Φ is the fundamental solution of heat equation. We claim that $u = \Phi_c * g$ solves the equation

$$\begin{cases} \partial_t u - \Delta u + cu = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{14.2}$$

We only need to verify u is continuous on the boundary. Notice that

$$\lim_{(x,t)\to(x_0,0)} \Phi * g(x,t) = g(x_0, 0)$$
$$\lim_{(x,t)\to(x_0,0)} u(x,t) = \lim_{(x,t)\to(x_0,0)} e^{-ct}\Phi * g(x,t) = g(x_0, 0)$$

As for the inhomogeneous equation, we apply the Duhamel's Principle, to get

$$u(x,t) = \int_0^t \Phi_c(\cdot, t-s) * f(\cdot, s)\, ds = \int_0^t \int_{\mathbb{R}^n} \Phi_c(x-y, t-s) f(y,s)\, dyds$$

which solves $\partial_t u - \Delta u + cu = f$ with zero initial data. Combine them together we get an explicit formula for (14.1):

$$u(x,t) = \int_{\mathbb{R}^n} \Phi_c(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi_c(x-y, t-s) f(y,s)\, dyds \tag{14.3}$$

with Φ_c defined as (14.2). □

15.[Ex.15] Given $g : [0,\infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)\ ds \tag{15.1}$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0,\infty). \end{cases}$$

(Hint: Let $v(x,t) \coloneqq u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.)

Proof. Let $v(x,t)$ be defined as the hint. Then $v = 0$ on $\{x = 0\} \times [0,\infty)$. And $v = 0$ on $\{t = 0\}$ since $g(0) = u(0,0) = 0$. Extend v to $\tilde{v}$ in $\mathbb{R} \times [0,\infty)$ by odd reflection. Then $\tilde{v}$ satisfies

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = f & \text{in } \mathbb{R} \times (0,\infty) \\ \tilde{v} = 0 & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{15.2}$$

where f is given by

$$f(x,t) = \begin{cases} -g'(t), & x > 0 \\ g'(t), & x < 0 \end{cases}$$

We could solve for $x, t > 0$:

$$\begin{aligned}
\tilde{v}(x,t) &= \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} f(y,s)\ dyds \\
&= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \Big(\int_{y>0} e^{-\frac{(x-y)^2}{4(t-s)}} (-g'(s))\ dy + \int_{y<0} e^{-\frac{(x-y)^2}{4(t-s)}} g'(s)\ dy \Big) ds \\
&= -\int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \Big(\int_{0<y<2x} e^{-\frac{(x-y)^2}{4(t-s)}} g'(s)\ dy \Big) ds \\
&= -\int_0^t g'(s) \Big(\frac{1}{\sqrt{\pi}} \int_{|y|<x/(2\sqrt{t-s})} e^{-y^2} dy \Big) ds \\
&= -g(t) + \int_0^t g(s) \frac{d}{ds} \Big(\frac{1}{\sqrt{\pi}} \int_{|y|<x/(2\sqrt{t-s})} e^{-y^2} dy \Big) ds \\
&= -g(t) + \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)\ ds
\end{aligned}$$

Thus, $u(x,t) = \tilde{v}(x,t) + g(t)$, we conclude. □

16.[Ex.16] Given a direct proof that if U is bounded and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation, then

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

(Hint: Define $u_\varepsilon := u - \varepsilon t$ for $\varepsilon > 0$, and show u_ε cannot attain its maximum over $\bar{U}_T$ at a point in U_T.)

Proof. Define u_ε as hint does. Then u_ε satisfies

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = -\varepsilon < 0$$

If u_ε attain its maximum at (x_0, t_0) in U_T. Then we have $\partial_t u_\varepsilon(x_0, t_0) \geq 0$, $\Delta u(x_0, t_0) \leq 0$. That's contradiction to $\partial_t u_\varepsilon - \Delta u_\varepsilon < 0$ in U_T. Thus

$$\max_{\bar{U}_T} u_\varepsilon = \max_{\Gamma_T} u_\varepsilon$$

Now we have

$$\max_{\bar{U}_T} u \leq \max_{\bar{U}_T} u_\varepsilon + \varepsilon T = \max_{\Gamma_T} u_\varepsilon + \varepsilon T \leq \max_{\Gamma_T} u + \varepsilon T$$

Let $\varepsilon \to 0$ we get $\max_{\bar{U}_T} u \leq \max_{\Gamma_T} u$, the other side follows immediately by $\Gamma_T \subset \bar{U}_T$ □

17.[Ex.17] We say $v \in C_1^2(U_T)$ is a *subsolution* of the heat equation if

$$v_t - \Delta v \leq 0 \qquad \text{in } U_T.$$

(a) Prove for a subsolution v that

$$v(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2} dyds$$

for all $E(x,t;r) \subset U_T$.

(b) Prove that therefore $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u solves the heat equation and $v := \phi(u)$. Prove that v is a subsolution.

(d) Prove $v := |Du|^2 + u_t^2$ is a subsolution, whenever u solves the heat equation.

Proof. For (c):

$$v_t - \Delta v = \phi'(u)u_t - \phi'(u)\Delta u - \phi''(u)(\nabla u)^2 = -\phi''(u)(\nabla u)^2 \leq 0$$

For (d):

$$\begin{aligned} v_t - \Delta v &= 2Du \cdot D(u_t - \Delta u) + 2u_t\partial_t(u_t - \Delta u) - 2|D^2u|^2 - 2|Du_t|^2 \\ &= -2(|D^2u|^2 + |Du_t|^2) \leq 0 \end{aligned}$$

□

18.[Ex.18] Assume u solves the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u = 0,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Show that $v := u_t$ solves:

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ v = h,\ v_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{18.1}$$

This is *Stokes' rule*

Proof. In $\mathbb{R}^n \times (0, +\infty)$ we have:

$$v_{tt} - \Delta v = \partial_{tt}(u_t) - \Delta(u_t) = \partial_t(u_{tt} - \Delta u) = 0$$

As for the boundary $\mathbb{R} \times \{t = 0\}$:

$$\begin{aligned} v(x, 0) &= u_t(x, 0) = h(x) \\ v_t(x, 0) &= u_{tt}(x, 0) = \Delta u(x, 0) = 0 \end{aligned}$$

whenever $u \in \overline{C^2(\mathbb{R}^n \times \{t \geq 0\})}$
Thus v solves equation (18.1). □

19.[Ex.19]

(a). Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x, y) = F(x) + G(y)$$

for arbitrary functions F and G.

(b). Using the change of variable $\xi = x + t,\ \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$

(c). Use (a) and (b) to rederive d'Alembert's formula.

(d). Under that condition on the initial data g, h is the solution of u a right-moving wave? A left-moving wave?

Proof. (a). Consider PDE of $\partial_y(u_x) = 0$, we have

$$u_x(x, y) = u_x(x, 0) + \int_0^y \partial_y(u_x(x, s))ds = u_x(x, 0)$$

We define $f(x) \coloneqq u_x(x, 0)$. Integral it we get $u(x, y)$:

$$u(x, y) = u(0, y) + \int_0^x u_x(s, y)ds = u(0, y) + \int_0^x f(s)ds \tag{19.1}$$

Define $F(x) \coloneqq \int_0^x f(s)ds$, $G(y) = u(0, y)$. Formula (19.1) becomes

$$u(x, y) = F(x) + G(y) \tag{19.2}$$

Thus a necessary condition for $u_{xy} = 0$ is $u = F(x) + G(y)$. Now we show that it is sufficient:

$$\partial_{xy}(F(x) + G(y)) = 0$$

Hence, the general solutions to $u_{xy} = 0$ is $u = F(x) + G(y)$ for arbitrary C^1 functions F, G.

(b). Direct computation shows:

$$\begin{aligned} u_{\eta\xi} &= \partial_\xi(u_x \frac{\partial x}{\partial \eta} + u_t \frac{\partial t}{\partial \eta}) = \frac{1}{2}\partial_\xi(u_x - u_t) = \frac{1}{2}(\partial_x - \partial_t)(\partial_\xi u) \\ &= \frac{1}{4}(\partial_x - \partial_t)(\partial_x + \partial_t)u = \frac{1}{4}u_{xx} - u_{tt} = 0 \end{aligned}$$

(c). By (a) and (b), if u is a solution to $u_{tt} - u_{xx} = 0$, then $u_{\eta\xi} = 0$ and

$$u(x,t) = F(\xi) + G(\eta) = F(x+t) + G(x-t)$$

Boundary conditions show that F and G must satisfy:

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases}$$

we have solutions given by

$$F(x) = \frac{1}{2}(g(x) + \int_0^x h(s)ds) + C$$
$$G(x) = \frac{1}{2}(g(x) - \int_0^x h(s)ds) - C$$

where C is a constant. Thus u is given by

$$u(x,t) = F(x+t) + G(x-t) = \frac{1}{2}\Big(g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(s)ds\Big) \tag{19.3}$$

which is d'Alembert's formula.

(d). To make u left-moving, we only need to make $G \equiv 0$, i.e.

$$g(x) = \int_0^x h(s)ds + C$$

For some constant C. It is equivalent to $g' = h$. Then $u(x,t) = F(x+t)$ is left-moving wave.

Such a condition is also necessary. If u is left-moving, that is

$$u(x,t) = u(x+t,0) \quad \text{for } \forall x \in \mathbb{R},\ t > 0$$
$$\iff G(x,t) = G(x-t) \quad \text{for } \forall x \in \mathbb{R},\ t > 0$$
$$\iff G \equiv constant \iff g' = h$$

Thus u is a left-moving wave if and only if $g' = h$.

By the same reason, u is right-moving if and only if $g' = -h$.

□

20.[Ex.21]

(a) Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ solve Maxwell's equations

$$\begin{cases} \mathbf{E}_t = \text{curl } \mathbf{B}, \quad \mathbf{B}_t = -\text{curl } \mathbf{E} \\ \text{div } \mathbf{B} = \text{div } \mathbf{E} = 0 \end{cases} \tag{20.1}$$

Show

$$\mathbf{E}_{tt} - \Delta \mathbf{E} = 0, \quad \mathbf{B}_{tt} - \Delta \mathbf{B} = 0$$

(b) Assume that $\mathbf{u} = (u^1, u^2, u^3)$ solves the evolution equations of linear elasticity

$$\mathbf{u}_{tt} - \mu \Delta \mathbf{u} - (\lambda + \mu) D(\text{div } \mathbf{u}) = \mathbf{0} \quad \text{in } \mathbb{R}^3 \times (0, \infty).$$

Show $w := \text{div } \mathbf{u}$ and $\mathbf{w} := \text{curl } \mathbf{u}$ each solve wave equations, but with differing speeds of propagation.

Proof. For (a): Notice

$$\text{curl curl } \mathbf{E} = \text{curl } \begin{pmatrix} \partial_2 E^3 - \partial_3 E^2 \\ \partial_3 E^1 - \partial_1 E^3 \\ \partial_1 E^2 - \partial_2 E^1 \end{pmatrix} = -\Delta \mathbf{E} + \nabla(\text{div } \mathbf{E})$$

Thus

$$\mathbf{E}_{tt} = \partial_t(\text{curl } \mathbf{B}) = \text{curl } \mathbf{B}_t = -\text{curl curl } \mathbf{E} = \Delta \mathbf{E}$$

The same for $\mathbf{B}$.
For (b):

$$\begin{aligned} w_{tt} &= \text{div } \mathbf{u}_{tt} = \text{div } \Big(\mu \Delta \mathbf{u} + (\lambda + \mu) D(\text{div } \mathbf{u})\Big) \\ &= (\lambda + 2\mu)\Delta(\text{div } \mathbf{u}) = (\lambda + 2\mu)\Delta w \end{aligned}$$

Now notice that curl $\nabla f = 0$ for $\forall f \in C^2$.

$$\mathbf{w}_{tt} = \text{curl } \mathbf{u}_{tt} = \text{curl } \Big(\mu \Delta \mathbf{u} + (\lambda + \mu) D(\text{div } \mathbf{u})\Big) = \mu \Delta \mathbf{w}$$

Thus $w, \mathbf{w}$ satisfy wave equations with propagation speed of $\lambda + 2\mu$ and μ. $\square$

21.[Ex.22] Let u denote the density of particles moving to the right with speed one along the real line and let v denote the density of particles moving to the left with speed one. If at rate $d > 0$ right-moving particles randomly become leaft-moving, and vice versa, we have the system of PDE

$$\begin{cases} u_t + u_x = d(v-u) \\ v_t - v_x = d(u-v) \end{cases}$$

Show that both $w := u$ and $w := v$ solve the telegraph equation

$$w_{tt} + 2dw_t - w_{xx} = 0$$

Proof. Let's suppose $w = u$ without loss of generality. Then

$$\begin{aligned} w_{tt} = u_{tt} &= d(v_t - u_t) - u_{tx} \\ w_{xx} = u_{xx} &= d(v_x - u_x) - u_{tx} \end{aligned}$$

Then we have

$$\begin{aligned} w_{tt} - w_{xx} &= d(v_t - v_x) - d(u_t - u_x) \\ &= d^2(u-v) + d^2(v-u) - 2du_t \\ &= -2du_t = -2dw_t \end{aligned}$$

By symmetry we have v also satisfies the telegraph equation. □

22.[Ex.23] Let S denote the square lying in $\mathbb{R} \times (0, \infty)$ with corners at the points $(0, 1)$, $(1, 2)$, $(0, 3)$. $(-1, 2)$. Define

$$f(x,t) := \begin{cases} -1 & \text{for } (x,t) \in S \cap \{t > x+2\} \\ 1 & \text{for } (x,t) \in S \cap \{t < x+2\} \\ 0, & \text{otherwise.} \end{cases}$$

Assume u solves

$$\begin{cases} u_{tt} - u_{xx} = f & \text{in } \mathbb{R} \times (0,\infty) \\ u = 0, \ u_t = 0 & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

Describe the shape of u for times $t > 3$.
(J.G Kingston, SIAM Review 30 (1988), 645-649)

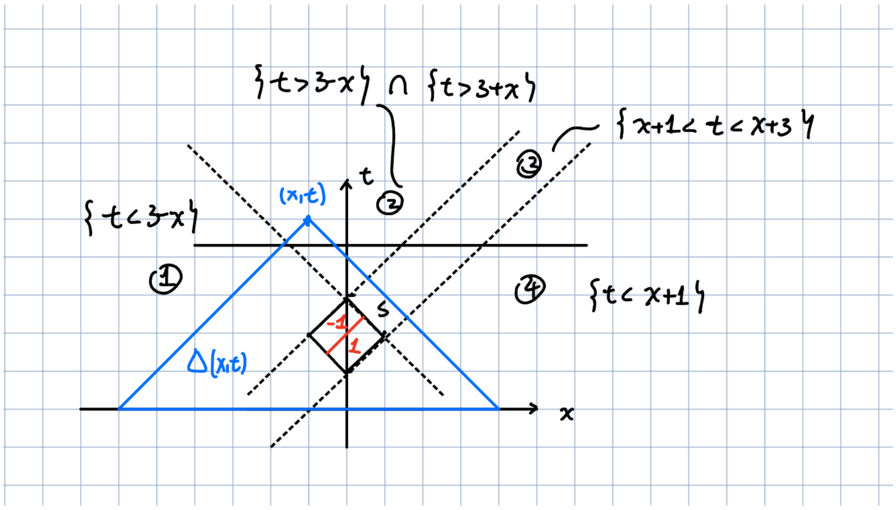

Figure 1: 22-1

By direct computation, Let $\Delta(x,t)$ be the triangle domain of $\{(y,s) : 0 < s < t,\ x - s < y < x + s\}$

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y, t-s)\ dyds = \frac{1}{2}\int_{\Delta(x,t)} f(y,s)\ dyds \\
&= \frac{1}{2}\int_{S\cap\Delta} f\ dyds \cdot \mathbf{1}_{\{t<3-x\}} + 0\cdot \mathbf{1}_{\{t>3-x\}\cap\{t>3+x\}} + 0\cdot \mathbf{1}_{\{t<x+1\}} + \\
&\quad \frac{1}{2}\int_{S\cap\Delta} f\ dyds \cdot \mathbf{1}_{\{x+1<t<x+3\}} \\
&= \begin{cases} t-x-1 & 1 < t-x < 2 \\ 3-t+s & 2 \le t-x \le 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

See figure 1 for the partition and $\Delta(x,t)$, and We could see $u(x,t)$ is a right-moving wave..

23.[Ex.24] (Equipartition of energy) Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Suppose g, h have compact support. The *kinetic energy* is $k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) \, dx$ and the *potential energy* is $p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) \, dx$. Prove

(a) $k(t) + p(t)$ is constant in t

(b) $k(t) = p(t)$ for all large enough times t.

Proof. For (a): By d'Alembert's formula, we have

$$u(x, t) = \frac{g(x - t) + g(x + t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(s) \, ds$$

Since g, h have compact support, $Du, \ D^2u$ have compact support and summable for all t. Hence, $k(t), \ p(t) \in C_0^1$.

$$\begin{aligned} \frac{d}{dt}(k(t) + p(t)) &= \int_{-\infty}^{+\infty} u_{tt} u_t \, dx + \int_{-\infty}^{+\infty} u_x u_{xt} \, dx \\ &= \int_{-\infty}^{+\infty} u_{xx} u_t + u_x u_{xt} \, dx \\ &= \int_{-\infty}^{+\infty} \frac{d}{dx}(u_x u_t) \, dx = 0 \end{aligned}$$

Thus, $k(t) + p(t)$ is a constant in t.
For (b): Use D'Amlerbert's formula, direct computation shows

$$k(t) - p(t) = \int_{-\infty}^{+\infty} -g'(x - t)g'(x + t) + h'(x - t)h'(x + t) \, dx$$

Since g, h are compactly supported, suppose $g \equiv h \equiv 0$ in $B(R)^c$. When $t > 2R$, one of $g'(x-t)$ or $g'(x+t)$ must lie out of supp g, which means $g'(x-t)g'(x+t) \equiv 0$. By the same reason, $h(x - t)h(x + t) \equiv 0$, thus

$$k(t) - p(t) = 0$$

That is, $k(t) = p(t)$ for $t > 2R$.

□