6 Chapter 6 Second order Elliptic PDE

1.[Ex.1] Consider Laplace's Equation with potential function c:

$$-\Delta u + cu = 0 \tag{1.1}$$

and the divergence structure equation:

$$-\mathrm{div}(aDv) = 0 \tag{1.2}$$

where the function a is positive

(a) Show that if u solves (1.1) and $w > 0$ solves (1.1), then $v := u/w$ solves (1.2) for $a := w^2$.

(b) Conversely, show that if v solves (1.2), then $u := va^{1/2}$ solves (1.1) for some potential c.

Proof. For (a): since u and w solves (1.1). $w > 0$

$$-\Delta u + cu = 0 \quad , \quad -\Delta w + cw = 0$$

Now $v = u/w \implies u = v \cdot w$.
Thus, $-\Delta(vw) + c \cdot vw = 0$

$$\begin{aligned}
&\iff -\Delta v \cdot w - 2\nabla v \cdot \nabla w - v \cdot \Delta w + c \cdot vw = 0 \\
&\iff -\Delta v \cdot w^2 - 2\nabla v \cdot \nabla w \cdot w = 0 \\
&\iff -\mathrm{div}(w^2 Dv) = 0
\end{aligned}$$

$$\begin{aligned}
\text{For (b):} \quad & \text{if } v \text{ solves (1.2)}, \quad u = v \cdot a^{1/2} \\
\Rightarrow \quad & -\mathrm{div}\left(aD\left(\frac{u}{a^{1/2}}\right)\right) = 0 \\
\iff \quad & -\mathrm{div}\left(a^{1/2}Du\right) + \mathrm{div}\left(\frac{u}{2a^{1/2}}Da\right) = 0 \\
\iff \quad & a^{1/2}\Delta u - \frac{1}{2a^{1/2}}\nabla a \cdot \nabla u + \frac{1}{2a^{1/2}}\nabla a \cdot \nabla u + \mathrm{div}\left(\frac{Da}{2a^{1/2}}\right)u = 0 \\
\iff \quad & -\Delta u + \frac{1}{a^{1/2}}\mathrm{div}\left(\frac{Da}{2a^{1/2}}\right)u = 0
\end{aligned}$$

□

2.[Ex.2] Let

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu \tag{2.1}$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided

$$c(x) \geq -\mu \quad (x \in U).$$

Proof. Define $B[\cdot,\cdot] : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ as

$$B[u,v] = \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + cuv$$

It is easy to see $B[\cdot,\cdot]$ is a bilinear form on U.
Since L is elliptic,

$$\begin{aligned} B[u,v] &= \int_U \sum a^{ij} u_{x_i} v_{x_j} + cuv \\ &\leq \Lambda \|Du\|_{L^2} \cdot \|Dv\|_{L^2} + \|c\|_\infty \|u\|_{L^2} \|v\|_{L^2} \\ &\leq (\|c\|_\infty + \Lambda) \|u\|_{H^1} \|v\|_{H^1} \end{aligned}$$

Thus, $B[\cdot,\cdot]$ is continuous. Moreover, assume $c(x) \geq -\mu$.
By Poincaré's inequality, there exists $C > 0$ such that

$$\|u\|_{H^1}^2 \leq C \|Du\|_{L^2}^2$$

Hence, we have

$$\begin{aligned} B[u,u] &\geq \lambda \|Du\|_{L^2}^2 + \int_U cu^2 \\ &\geq \lambda \|Du\|_{L^2}^2 - \mu \|u\|_{L^2}^2 \\ &\geq \left(\frac{\lambda}{C} - \mu \right) \|u\|_{H^1}^2 \end{aligned}$$

Let $0 < \mu < \frac{\lambda}{C}$, when $c(x) > \mu$,

$$B[u,u] \gtrsim \|u\|_{H^1}^2$$

This shows the coercivity of $B[\cdot,\cdot]$

$\square$

3.[Ex.3] A function $u \in H_0^2(U)$ is a weak solution of this boundary-value problem for the *biharmonic equation*

$$\begin{cases} -\Delta^2 u = f & \text{in } U \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \tag{3.1}$$

provided

$$\int_U \Delta u \Delta v \, dx = \int_U fv \, dx \tag{3.2}$$

for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove that there exists a unique weak solution of (3.1).

Proof. **Existence:** Define $B[\cdot,\cdot] : H_0^2(U) \times H_0^2(U) \to \mathbb{R}$ as

$$B[u,v] = \int_U \Delta u \cdot \Delta v \, dx$$

Obviously $B[\cdot,\cdot]$ is linear, continuous. By Poincaré's inequality, applying some integration by parts, we also have coercivity:

$$\begin{aligned} B(u,u) &= \int_U (\Delta u)^2 \, dx = \sum_{i,j=1}^n \int_U u_{x_i x_i} u_{x_j x_j} \, dx \\ &= \sum_{i,j=1}^n \int_U u_{x_i x_j} u_{x_i x_j} \, dx = \int_U |D^2 u|^2 \, dx \\ &\geq C\|u\|_{H_0^2}^2 \end{aligned}$$

Thus, the Lax–Milgram Theorem shows that for given $f \in L^2(U)$, there exists a solution $u \in H_0^2(U)$ to (3.1).

Uniqueness: Let u_1, u_2 both be solutions to (3.1).

Define $u = u_1 - u_2$, then u is a weak solution to

$$\begin{cases} -\Delta^2 u = 0 & \text{in } U \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

And

$$\|u\|_{H_0^2}^2 \lesssim \|D^2 u\|_{L^2}^2 = \int_U (\Delta u)^2 \, dx = \int_U 0 \cdot u \, dx = 0$$

$$\implies u \equiv 0$$

□

4.[Ex.4] Assume U is connected. A function $u \in H^1(U)$ is a weak solution of *Neumann's problem*

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \tag{4.1}$$

if

$$\int_U Du \cdot Dv \; dx = \int_U fv \; dx \tag{4.2}$$

for all $v \in H^1(U)$. Suppose $f \in L^2(U)$. Prove (4.1) has a weak solution if and only if

$$\int_U f \; dx = 0 \tag{4.3}$$

Proof. On the one hand, if u is a weak solution, let $v = 1$ in U. Then (4.2) gives

$$\int_U f\,dx = \int_U Du \cdot 0\,dx = 0.$$

On the other hand, if $\int_U f\,dx = 0$, let $M := \left\{f \in L^2 : \int_U f\,dx = 0\right\}, \quad M^1 := M \cap H^1$. Define $B[\cdot,\cdot] : M^1 \times M^1 \to \mathbb{R}$ as

$$B[u,v] = \int_U Du \cdot Dv\,dx$$

It is easy to verify $B[u,v]$ is bilinear and continuous. By Poincaré's inequality, in M^1, we have

$$\int_U u^2\,dx \le C \int_U |Du|^2\,dx.$$

Thus, $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax–Milgram *Theorem.

$$\Rightarrow \exists u \in M^1 \text{ such that for all } v \in M^1, \qquad \int_U Du \cdot Dv\,dx = \int_U fv\,dx.$$

To make u a weak solution, we need to take $v \in H^1$, not M^1. Let $v \in H^1$. Then $v - \int_U v\,dx \in M^1$.

$$\begin{aligned} \Rightarrow \int_U Du \cdot Dv\,dx &= \int_U Du \cdot D\left(v - \int_U v\,dx\right)\,dx \\ &= \int_U f\left(v - \int_U v\right)dx \\ &= \int_U fv - \int_U v \cdot \int_U f = \int_U fv. \end{aligned}$$

Thus, u is the weak solution.

□

5.[Ex.5] Explain how to define $u \in H^1(U)$ to be a weak solution of Poisson's equation with *Robin boundary conditions*:

$$\begin{cases} -\Delta u = f & \text{in } U \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \tag{5.1}$$

Discuss the existence and uniqueness of a weak solution for a given $f \in L^2(U)$.

Proof. Apply integration by parts,

$$\begin{aligned} \int_U fv\,dx = \int_U -\Delta u \cdot v\,dx &= \int_U \nabla u \cdot \nabla v\,dx - \int_{\partial U} \frac{\partial u}{\partial \nu} v\,dx \\ &= \int_U \nabla u \cdot \nabla v\,dx + \int_{\partial U} u \cdot v\,dx. \end{aligned}$$

Then it is reasonable to define $u \in H^1(U)$ as a weak solution if

$$\int_U \nabla u \cdot \nabla v\,dx + \int_{\partial U} u \cdot v\,dx = \int_U f \cdot v\,dx \quad \text{for all } v \in H^1(U).$$

Define

$$B[u,v] = \int_U \nabla u \cdot \nabla v\,dx + \int_{\partial U} u \cdot v\,dx.$$

Then $B[\cdot,\cdot]$ is bilinear and bounded, since we have the trace inequality

$$\int_{\partial U} |u|^2\,ds \le C\|u\|_{H^1}^2.$$

And Poincaré's inequality,

$$\int_U |u|^2\,dx \le C\left(\int_U |\nabla u|^2\,dx + \int_{\partial U} |u|^2\,ds\right)$$

shows the coercivity of $B[\cdot,\cdot]$. Thus, we can apply the Lax–Milgram Theorem to deduce the existence of a weak solution.
For uniqueness, assume u_1 and u_2 are two weak solutions. Let $u = u_1 - u_2$. Then,

$$\int_U |\nabla u|^2\,dx + \int_{\partial U} |u|^2\,ds = \int_U 0 \cdot u\,dx = 0.$$

Thus, $\nabla u \equiv 0$, $u|_{\partial U} \equiv 0 \implies u = 0$.

□

6.[Ex.6] Suppose U is connected and ∂U consists of two disjoint, closed sets Γ_1 and Γ_2. Define what if means for u to be a weak solution of Poisson's equation with *mixed Dirichlet-Neumann Boundary Conditions*:

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2 \end{cases} \tag{6.1}$$

Discuss the existence and uniqueness of weak solutions.

Proof. Define

$$H(U) = \left\{ u \in H^1(U) : u = 0 \text{ on } \Gamma_1 \right\},$$

it is easy to see $H(U)$ is still a Hilbert space.
A weak solution to (6.1) is defined as $u \in H(U)$ such that

$$\int_U fv\,dx = \int_U \nabla u \cdot \nabla v\,dx \quad \text{for all } v \in H(U).$$

By the standard Lax–Milgram Theorem, notice

$$B[u,v] := \int_U \nabla u \cdot \nabla v\,dx$$

is bilinear, continuous, and coercive. Since Poincaré's inequality works in H, we deduce the existence.
For uniqueness, it is also standard as we did in [Ex.5].

□

7.[Ex.7] Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n, \tag{7.1}$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$.
(Hint: Mimic the proof of Theorem 1 in §6.3.1, but without the cutoff function ζ.)

Proof. Since u is a weak solution, we have

$$\int_U \nabla u \cdot \nabla v + c(u) \cdot v \, dx = \int_U f \cdot v \quad \text{for all } v \in H^1(\mathbb{R}^n).$$

Take $v = D_{-h}(D_h u) \in H^1(\mathbb{R}^n)$.

$$\Rightarrow \int |\nabla(D_h u)|^2 + D_h(c(u)) \cdot D_h u = \int f \cdot D_{-h} D_h u \, dx.$$

Now since $c' \geq 0 \Rightarrow c$ is increasing, $D_h(c(u)) \cdot D_h u \geq 0$.

$$\begin{aligned} \Rightarrow \int |\nabla(D_h u)|^2 &\leq \int f \cdot D_{-h} D_h u \, dx \\ &\leq \|f\|_{L^2} \cdot \|D_{-h} D_h u\|_{L^2} \\ &\leq \|f\|_{L^2} \cdot \|\nabla D_h u\|_{L^2} \end{aligned}$$

Thus we have

$$\|D_h(\nabla u)\|_{L^2} = \|\nabla(D_h u)\|_{L^2} \leq \|f\|_{L^2} \quad \text{for all } h \in \mathbb{R}^n.$$

Let $h \to 0$. This implies $\nabla u \in H^1(\mathbb{R}^n)$, i.e., $u \in H^2(\mathbb{R}^n)$, with $\|\nabla^2 u\|_{L^2} \leq \|f\|_{L^2}$. $\square$

8.[Ex.8] Let u be a smooth solution of the uniformly elliptic equation $Lu = -\sum_{i,j=1}^n a^{ij}(x)u_{x_ix_j} = 0$ in U. Assume that the coefficients have bounded derivatives.
Set $v := |Du|^2 + \lambda u^2$ and show that

$$Lv \leq 0 \quad \text{in} U \tag{8.1}$$

if λ is large enough. Deduce

$$\|Du\|_{L^\infty(U)} \leq C\Big(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}\Big) \tag{8.2}$$

Proof. Firstly, since

$$-\sum_{i,j} a^{ij}u_{x_ix_j} = 0,$$

differentiate both sides, we have

$$-\sum_{i,j} D(a^{ij})u_{x_ix_j} = \sum_{i,j} a^{ij}(Du)_{x_ix_j}.$$

Now assume $|Da^{ij}| \leq M$. And a^{ij} is uniformly elliptic:

$$\sum_{i,j} a^{ij}\xi_i\xi_j \geq \Lambda|\xi|^2.$$

Since $v = |Du|^2 + \lambda u^2$, we have

$$\begin{aligned}
Lv &= -\sum_{i,j=1}^n a^{ij}(x) \cdot 2\left((Du)_{x_i}(Du)_{x_j} + (Du)_{x_ix_j} \cdot Du + \lambda u_{x_i}u_{x_j} + \lambda u u_{x_ix_j}\right) \\
&= -2\sum_{i,j=1}^n a^{ij}(x)\left((Du)_{x_i}(Du)_{x_j} + \lambda u_{x_i}u_{x_j}\right) + 2\sum_{i,j} D(a^{ij}(x))u_{x_ix_j} \cdot Du \\
&\leq -2\Lambda|D^2u|^2 - 2\lambda\Lambda|Du|^2 + \frac{M}{\Lambda}|Du|^2 + \Lambda|D^2u|^2 \\
&\leq -\Lambda|D^2u|^2 - \left(2\lambda\Lambda - \frac{M}{\Lambda}\right)|Du|^2 \leq 0 \quad \text{when } \lambda \geq \frac{M}{2\Lambda^2}.
\end{aligned}$$

Now $Lv \leq 0$, by the maximum principle: $\sup_U |v| \leq \sup_{\partial U} |v|$

$$\Rightarrow \|Du\|_{L^\infty}^2 \leq \sup_U |v| \leq \sup_{\partial U} |Du|^2 + \lambda \sup_{\partial U} |u|^2$$

$\square$

9.[Ex.9] Assume u is a smooth solution of $Lu = -\sum_{i,j=1}^n a^{ij}u_{x_ix_j} = f$ in U, $u = 0$ on ∂U, where f is bounded. Fix $x^0 \in \partial U$. A *barrier* at x^0 is a C^2 function w such that

$$Lw \geq 1 \text{ in } U, \quad w(x^0) = 0, \quad w \geq 0 \text{ on } \partial U. \tag{9.1}$$

Show that if w is a barrier at x^0, there exists a constant C such that

$$|Du(x^0)| \leq C \left|\frac{\partial w}{\partial \nu}(x^0)\right|.$$

Proof. Assume

$$\sup_U f_+ = M.$$

Then

$$L(u - Mw) = f - M \leq 0,$$

By the Maximum Principle, $u - Mw \leq 0$ in U.
Since $u = 0$ on ∂U,

$$|Du(x_0)| = \frac{\partial u}{\partial \nu}(x_0).$$

And,

$$u - Mw(x_0) = 0 = \sup_U (u - Mw).$$

By Hopf's lemma,

$$\frac{\partial}{\partial \nu}(u - Mw) \leq 0 \implies \frac{\partial u}{\partial \nu} \leq M \cdot \frac{\partial w}{\partial \nu}.$$

□

10.[Ex.10] Assume U is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

are $u \equiv C$, for some constant C.

Proof. **(a) Energy methods.**

$$0 = \int_U (-\Delta u) \cdot u = \int_U |\nabla u|^2 + \int_{\partial U} \frac{\partial u}{\partial \nu} \cdot u = \int_U |\nabla u|^2$$

Thus, $\nabla u \equiv 0$ a.e., which implies $u \equiv C$ for some constant C.

(b) Maximum Principle.

$$-\Delta u \leq 0 \quad \Rightarrow \quad u \text{ satisfies the maximum principle.}$$

Suppose u is not constant, and $u(x_0) = \max_{\partial U} |u|$ with $x_0 \in \partial U$. Then by Hopf's lemma,

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

That's a contradiction to

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial U} = 0.$$

Thus, $u \equiv C$ is a constant function.

□

11.[Ex.11] Assume $u \in H^1(U)$ is a bounded weak solution of

$$-\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} = 0 \quad \text{in } U.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex and smooth, and set $w = \phi(u)$. Show w is a weak subsolution; that is, $B[w,v] \leq 0$ for all $v \in H_0^1(U)$, $v \geq 0$.

Proof. Take $v \in C_c^\infty(U)$,

$$\begin{aligned}
B[w,v] &= \int \sum_{i,j=1}^{n} a^{ij} \left(\phi(u)\right)_{x_i} v_{x_j} \, dx \\
&= \int \sum_{i,j=1}^{n} a^{ij} \phi'(u) u_{x_i} v_{x_j} \, dx \\
&= \int \sum_{i,j=1}^{n} a^{ij} u_{x_i} \left(\phi'(u) v\right)_{x_j} \, dx - \int \sum_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j} \phi''(u) v \, dx \\
&\leq \int \sum_{i,j=1}^{n} a^{ij} u_{x_i} \left(\phi(u) v\right)_{x_j} \, dx = 0
\end{aligned}$$

Since $C_c^\infty(U)$ is dense in $H_0^1(U)$, and $B[\cdot,\cdot]$ is continuous, the inequality holds for all $v \in H_0^1$. Thus, w is a weak subsolution.

□

12.[Ex.12] We say that the uniformly elliptic operator

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu$$

satisfies the weak maximum principle if for all $u \in C^2(U) \cap C(\overline{U})$

$$\begin{cases} Lu \le 0 & \text{in } U \\ u \le 0 & \text{on } \partial U \end{cases}$$

implies that $u \le 0$ in U.
Suppose that there exists a function $v \in C^2(U) \cap C(\overline{U})$ such that $Lv \ge 0$ in U and $v > 0$ on $\overline{U}$. Show that L satisfies the weak maximum principle.
(Hint: Find an elliptic operator M with no zeroth-order term such that $w := u/v$ satisfies $Mw \le 0$ in the region $\{u > 0\}$. To do this, first compute $(v^2 w_{x_i})_{x_j}$.)

Proof. Let $w := \frac{u}{v}$. Then $w > 0$ on $\{u > 0\}$, and $w = 0$ on ∂U.

$$L(w \cdot v) = Lu \le 0$$

$$\begin{aligned}
&\implies \quad -\sum_{i,j} a^{ij} w_{x_i x_j} \cdot v - 2\sum_{i,j} a^{ij} w_{x_i} v_{x_j} + Lv \cdot w + \sum_{i=1}^{n} b^i w_{x_i} \cdot v \le 0 \\
&\implies \quad -\sum_{i,j} a^{ij} w_{x_i x_j} + \sum_{i=1}^{n} \left(b^i - \sum_{j=1}^{n} 2a^{ij} \frac{v_{x_j}}{v} \right) w_{x_i} \le -\frac{Lv}{v} \cdot w \le 0 \quad \text{in } \{u > 0\}
\end{aligned}$$

Define

$$Mw := -\sum_{i,j=1}^{n} a^{ij} w_{x_i x_j} + \sum_{i=1}^{n} \left(b^i - \sum_{j=1}^{n} 2a^{ij} \frac{v_{x_j}}{v} \right) w_{x_i}.$$

Then M has no zero-th order term. By the maximum principle,

$$\sup_U w \le \sup_{\partial U} w^+ = 0 \quad \implies \quad w \le 0 \text{ in } U \quad \implies \quad u \le 0 \text{ in } U.$$

□

13.[Ex.13] (Courant minimax principle) Let $L = -\sum_{i,j=1}^{n}(a^{ij}u_{x_i})_{x_j}$, where (a^{ij}) is symmetric. Assume the operator L, with zero boundary conditions, has eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots$. Show

$$\lambda_k = \max_{S\in\Sigma_{k-1}} \min_{\substack{u\in S^\perp \\ \|u\|_{L^2}=1}} B[u,u] \quad (k = 1, 2, \dots).$$

Here Σ_{k-1} denotes the collection of $(k-1)$-dimensional subspaces of $H_0^1(U)$.

Proof. Let $\{w_n\}_{n=1}^\infty$ be the eigenfunctions with respect to $\{\lambda_k\}_{k=1}^\infty$, and $\|w_n\|_{L^2} = 1$. Then $\{w_n\}_{n=1}^\infty$ forms an orthonormal basis in $L^2(U)$.
Let $S_0 := \operatorname{span}\{w_1, \dots, w_{k-1}\}$ which has dimension $k-1$, and $w_k \in S_0^\perp$.

$$B[w_k, w_k] = (Lw_k, w_k) = \lambda_k.$$

And for $u \in S_0^\perp$, let

$$u = \sum_{n=0}^{\infty} d_{k+n}w_{k+n}, \quad \text{with} \sum_{n=0}^{\infty} d_{k+n}^2 = 1.$$

Then, we calculate

$$B[u,u] = \sum_{n=0}^{\infty} d_{k+n}^2\lambda_{k+n} \ge \lambda_k \sum_{n=0}^{\infty} d_{k+n}^2 = \lambda_k.$$

Thus,

$$\min_{u\in S_0^\perp} B[u,u] = \lambda_k \le \max_{S\in\Sigma_{k-1}} \min_{u\in S^\perp} B[u,u]. \tag{13.1}$$

On the other hand, let $S_1 := \operatorname{span}\{w_1, \dots, w_k\}$ which has dimension k. Thus, for all $S \in \Sigma_k$, we could find $u \in S^\perp \cap S_1$. Let $u = \sum_{n=1}^{k} d_n w_n$, then

$$B[u,u] = \sum_{n=1}^{k} d_n^2\lambda_n \le \lambda_k \quad \Rightarrow \quad \min_{S^\perp} B[u,u] \le \lambda_k.$$

Thus,

$$\max_{S\in\Sigma_{k-1}} \min_{u\in S^\perp} B[u,u] \le \lambda_k. \tag{13.2}$$

Combine (13.1) and (13.2), we conclude with

$$\lambda_k = \max_{S\in\Sigma_{k-1}} \min_{u\in S^\perp} B[u,u].$$

□

14.[Ex.14] Let λ_1 be the principal eigenvalue of the uniformly elliptic, nonsymmetric operator

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu,$$

taken with zero boundary conditions. Prove the "max-min" representation formula:

$$\lambda_1 = \sup_u \inf_{x \in U} \frac{Lu(x)}{u(x)},$$

the "sup" taken over functions $u \in C^\infty(\overline{U})$ with $u > 0$ in U, $u = 0$ on ∂U, and the "inf" taken over points $x \in U$.
(Hint: Consider the eigenfunction w_1^* corresponding to λ_1 for the adjoint operator L^*.)

Proof. Let w_1 be the eigenfunction of L corresponding to λ_1.
Then $w_1 > 0$ and smooth. Thus,

$$\inf_{x \in U} \frac{Lw_1(x)}{w_1(x)} = \lambda_1.$$

$$\Longrightarrow \lambda_1 \le \sup_{\substack{u \in C_c^\infty(U) \\ u > 0}} \inf_{x \in U} \frac{Lu(x)}{u(x)} \tag{14.1}$$

On the other hand, consider w_1^* for the adjoint operator L^*.
Then

$$\begin{aligned}\int_U \frac{Lu(x)}{u(x)} u(x) w_1^*(x)\, dx &= \int_U Lu \cdot w_1^*\, dx = \int_U u \cdot L^* w_1^*\, dx \\ &= \lambda_1 \int_U u(x) w_1^*(x)\, dx\end{aligned}$$

$$\Longrightarrow \int_U \left(\frac{Lu(x)}{u(x)} - \lambda_1 \right) u(x) w_1^*(x)\, dx = 0$$

Since $u(x) w_1^*(x) > 0$, we deduce

$$\inf_{x \in U} \left(\frac{Lu(x)}{u(x)} - \lambda_1 \right) \le 0 \quad \text{for all } u > 0,\ u \in C^\infty(U)$$

$$\Longrightarrow \sup_u \inf_{x \in U} \frac{Lu(x)}{u(x)} \le \lambda_1 \tag{14.2}$$

Combine (14.1) and (14.2), we conclude. □

15.[Ex.15] (Eigenvalues and domain variations) Consider a family of smooth, bounded domains $U(\tau) \subset \mathbb{R}^n$ that depend smoothly upon the parameter $\tau \in \mathbb{R}$. As τ changes, each point on $\partial U(\tau)$ moves with velocity $\mathbf{v}$.
For each τ, we consider eigenvalues $\lambda = \lambda(\tau)$ and corresponding eigenfunctions $w = w(x, \tau)$:

$$\begin{cases} -\Delta w = \lambda w & \text{in } U(\tau) \\ w = 0 & \text{on } \partial U(\tau), \end{cases}$$

normalized so that $\|w\|_{L^2(U(\tau))} = 1$. Suppose that λ and w are smooth functions of τ and x. Prove Hadamard's variational formula

$$\dot{\lambda} = -\int_{\partial U(\tau)} \left| \frac{\partial w}{\partial \nu} \right|^2 \mathbf{v} \cdot \nu \, dS,$$

where $\dot{\lambda} = \frac{d}{d\tau}\lambda$ and $\mathbf{v} \cdot \nu$ is the normal velocity of $\partial U(\tau)$.
(Hint: Use the calculus formula from §C.4.)

Proof. Since $w(x, \tau)$ is a weak solution, we have

$$\lambda = \int_U \lambda w \cdot w \, dx = \int_U |\nabla w|^2 \, dx.$$

Differentiate with respect to τ, noticing $|\nabla w|\big|_{\partial U} = \left|\frac{\partial w}{\partial \nu}\right|$, and using calculus formulas in §C.4, we get

$$\begin{aligned} \dot{\lambda} &= \int_{\partial U} |\nabla w|^2 \, \mathbf{v} \cdot \nu \, ds + \int_U \frac{d}{d\tau} |\nabla w|^2 \, dx \\ &= \int_{\partial U} \left(\frac{\partial w}{\partial \nu} \right)^2 \mathbf{v} \cdot \nu \, ds + \int_U \nabla w \cdot \nabla \dot{w} \, dx. \end{aligned}$$

And since $\|w\|_{L^2} \equiv 1$, we have

$$\int_U \nabla w \cdot \nabla \dot{w} = \int_U \lambda w \cdot \dot{w} = \lambda \cdot \frac{1}{2} \cdot \frac{d}{d\tau} \|w\|_{L^2}^2 = 0.$$

Thus,

$$\dot{\lambda} = \int_{\partial U} \left(\frac{\partial w}{\partial \nu} \right)^2 \mathbf{v} \cdot \nu \, ds.$$

□

16.[Ex.16] (Radiation condition) If we separate variables to look for a complex-valued solution of the wave equation having the form $u = e^{-i\sigma t}w$ for $w = w(x)$ and $\sigma \in \mathbb{R}, \sigma \neq 0$, we are led to the eigenvalue problem

$$(*) \qquad -\Delta w = \lambda w \quad \text{in } \mathbb{R}^n$$

where $\lambda := \sigma^2$.

(a) Show that $w = e^{i\sigma\omega\cdot x}$ solves $(*)$, provided $|\omega| = 1$. Then $u = e^{i\sigma(\omega\cdot x - t)}$ is a traveling wave function of the wave equation.

(b) Show that for $n = 3$, the function $\Phi := \frac{e^{i\sigma|x|}}{4\pi|x|}$ solves

$$-\Delta\Phi = \lambda\Phi + \delta_0 \quad \text{in } \mathbb{R}^3.$$

(c) The Sommerfeld radiation condition requires for a solution of $(*)$ that

$$\lim_{r\to\infty} r(w_r - i\sigma w) = 0,$$

for $w_r := Dw \cdot \frac{x}{|x|}$. Prove that the solution w from (a) does not satisfy this condition but that Φ from (b) does.

Proof. **(a).** Direct computation: $-\Delta w = -|\mathrm{i}\sigma\omega|^2 e^{\mathrm{i}\sigma\omega\cdot x} = \sigma^2 e^{\mathrm{i}\sigma\omega\cdot x} = \lambda w$. Then $u = e^{-\mathrm{i}\sigma t}w = e^{\mathrm{i}\sigma(\omega\cdot x - t)}$ solves the wave equation.

(b). We compute $\Delta\Phi$:

$$\begin{aligned}
-\Delta\Phi &= e^{\mathrm{i}\sigma|x|} \cdot -\Delta\left(\frac{1}{4\pi|x|}\right) - \Delta\left(e^{\mathrm{i}\sigma|x|}\right) \cdot \frac{1}{4\pi|x|} - 2\nabla\left(e^{\mathrm{i}\sigma|x|}\right) \cdot \nabla\left(\frac{1}{4\pi|x|}\right) \\
&= \delta_0 + \sigma^2 e^{\mathrm{i}\sigma|x|} \cdot \frac{1}{4\pi|x|} - \frac{2\mathrm{i}\sigma e^{\mathrm{i}\sigma|x|}}{4\pi|x|^2} + \frac{2\mathrm{i}\sigma e^{\mathrm{i}\sigma|x|}}{4\pi|x|^4} x \cdot x \\
&= \delta_0 + \lambda\Phi.
\end{aligned}$$

(c). Since we have $w_r - \mathrm{i}\sigma w = \left(\mathrm{i}\sigma\omega \cdot \frac{x}{|x|} - \mathrm{i}\sigma\right) e^{\mathrm{i}\sigma\omega\cdot x}$. If we choose $\omega, \frac{x}{|x|}$ such that $\omega \cdot \frac{x}{|x|} = 0$, then $w = 1$. $r(w_r - \mathrm{i}\sigma w) = -\mathrm{i}\sigma r$ does not converge to 0.

$$|\Phi_r - \mathrm{i}\sigma\Phi| = \left|\frac{e^{\mathrm{i}\sigma|x|}}{4\pi|x|^2}\right| \leq \frac{1}{4\pi r^2} \implies \lim_{r\to\infty} r\,|\Phi_r - \mathrm{i}\sigma\Phi| = 0.$$

□

17.[Ex.17] (Continuation) Prove that if w is a complex-valued solution of eigenvalue problem $(*)$ in $\mathbb{R}^3$ and if w satisfies the radiation condition, then $w \equiv 0$. (Hints: First observe that

$$0 = \int_{B(0,R)} (\overline{w}\Delta w - w\Delta\overline{w})\, dx = \int_{\partial B(0,R)} (\overline{w}w_r - w\overline{w}_r)\, dS.$$

Use this and the radiation condition to show

$$\int_{\partial B(0,R)} |w_r|^2 + \sigma^2|w|^2\, dS = \int_{\partial B(0,R)} |w_r - i\sigma w|^2\, dS \to 0$$

as $R \to \infty$. Given now a point $x_0 \in \mathbb{R}^3$, select $R > |x_0|$. Then

$$w(x_0) = \int_{\partial B(0,R)} (\Phi w_r - w\Phi_r)\, dS,$$

where $\Phi = \Phi(x - x_0)$. Show the integral goes to zero as $R \to \infty$.)

Proof. Follow the hints, firstly we observe

$$0 = \int_{B(0,R)} (\overline{w}\Delta w - w\Delta\overline{w})\, dx = \int_{\partial B(0,R)} (\overline{w}w_r - w\overline{w}_r)\, dS.$$

Then since $|w_r - i\sigma w|^2 = |w_r|^2 + |\sigma w|^2 - 2i(w_r\overline{w} - \overline{w}_r w)$

$$\int_{\partial B(0,R)} |w_r|^2 + \sigma^2|w|^2\, dS = \int_{\partial B(0,R)} |w_r - i\sigma w|^2\, dS \to 0$$

Now, we write $w(x_0)$ in forms of Φ as

$$\begin{aligned}
w(x_0) &= \int_{B(0,R)} \delta_{x_0} w \; dx = \int_{B(0,R)} (-\Delta\Phi - \lambda\Phi)\cdot w \\
&= \int_{B(0,R)} \Delta w \cdot \Phi - \Delta\Phi \cdot w \; dx = \int_{\partial B(0,R)} (\Phi w_r - w\Phi_r)\, dS \\
&\le \left(\frac{1}{\sigma}\int_{\partial B(0,R)} |w_r|^2 + \sigma^2|w|^2\, dS\right)^{1/2} \cdot \left(\frac{1}{\sigma}\int_{\partial B(0,R)} |\Phi_r|^2 + \sigma^2|\Phi|^2\, dS\right)^{1/2}
\end{aligned}$$

One could easily verify that

$$\left(\frac{1}{\sigma}\int_{\partial B(0,R)} |\Phi_r|^2 + \sigma^2|\Phi|^2\, dS\right)^{1/2} \le C \quad \text{for } \forall R > 0$$

Thus, since w satisfies the radiation condition, take $R \to \infty$ we conclude that $w(x_0) = 0$ for all $x_0 \in \mathbb{R}^3$ □