5 Sobolev Spaces

1.[Ex.1] Suppose $k \in \{0, 1, \dots\}$, $0 < \gamma \le 1$. Prove $C^{k,\gamma}(\bar{U})$ is a Banach space

Proof. Suppose $\{u_n\}_{n\in\mathbb{N}}$ is a Cauthy sequence in $C^{k,\gamma}(\bar{U})$. For $\forall\varepsilon > 0$, there exists $N \in \mathbb{N}$, when $n, m > N$ we have

$$\|\partial^\alpha u_n - \partial^\alpha u_m\|_{L^\infty} \le \|u_n - u_m\|_{C^{k,\gamma}} \le \varepsilon \tag{1.1}$$

for $\forall\alpha$ with $|\alpha| \le k$. Thus $\partial^\alpha u_n$ forms a Cauthy Sequence in L^∞. There exists a function g_α such that

$$\|\partial^\alpha u_n - g_\alpha\|_{L^\infty} \to 0$$

Especially assume u_n converges to u in L^∞. Since $\partial^\alpha u_n$ are continuous and uniformly converges to g_α, thus g_α is continuous. We claim that $u \in C^{k,\alpha}$ with $\partial^\alpha u = g_\alpha$. It suffices to prove $\partial_1 u = g_1$, others follow by induction on α.
Without loss of generality, assume $x = (x_1, x') \in B_r(0) \subset U$. Notice $u_n(x_1, x') = u_n(0, x') + \int_0^{x_1} \partial_1 u_n(s, x')\, ds$. We have

$$\begin{aligned} u(x) - \int_0^{x_1} g_1(s, x')ds &= \lim_{n\to\infty} \left(u_n(x) - \int_0^{x_1} \partial_1 u_n(s, x')\, ds\right) \\ &= \lim_{n\to\infty} u_n(0, x') = u(0, x') \end{aligned} \tag{1.2}$$

Thus $\partial_1 u(x) = g_1$, by induction we could see $u \in C^k(\bar{U})$. Now we show $\partial^\alpha u \in C^\gamma$ for $|\alpha| = k$. We know that

$$\begin{aligned} &[\partial^\alpha u_n]_\gamma \le \|u_n\|_{C^{k,\gamma}} \le M \\ \Longrightarrow\quad &|\partial^\alpha u_n(x) - \partial^\alpha u_n(y)| \le M\|x - y\|^\gamma \\ \Longrightarrow\quad &|\partial^\alpha u(x) - \partial^\alpha u(y)| = \lim_{n\to\infty} |\partial^\alpha u_n(x) - \partial^\alpha u_n(y)| \\ &\le M\|x - y\|^\gamma \text{ for } \forall x \ne y \end{aligned}$$

Hence, $u \in C^{k,\alpha}(\bar{U})$, with $\|u_n - u\|_{C^{k,\gamma}} \to 0$. That implies $C^{k,\gamma}(\bar{U})$ is a Banach Space □

2.[Ex.2] Assume $0 < \beta < \gamma \le 1$. Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(\bar{U})} \le \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \cdot \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}} \tag{2.1}$$

Proof. Notice that

$$\gamma = \beta \cdot \frac{1-\gamma}{1-\beta} + \frac{\gamma-\beta}{1-\beta}$$

Recall that we define $\|u\|_{C^{0,\gamma}}$ as

$$\begin{aligned}
\|u\|_{C^{0,\gamma}} &= \|u\|_{L^\infty} + \sup_{x\neq y} \frac{|u(x)-u(y)|}{\|x-y\|^\gamma} \\
&= \|u\|_{L^\infty}^{\frac{1-\gamma}{1-\beta}+\frac{\gamma-\beta}{1-\beta}} + \sup_{x\neq y} \frac{|u(x)-u(y)|^{\frac{1-\gamma}{1-\beta}+\frac{\gamma-\beta}{1-\beta}}}{\|x-y\|^{\beta\cdot\frac{1-\gamma}{1-\beta}+\frac{\gamma-\beta}{1-\beta}}} \\
&\le \|u\|_{L^\infty}^{\frac{1-\gamma}{1-\beta}} \cdot \|u\|_{L^\infty}^{\frac{\gamma-\beta}{1-\beta}} + \left(\sup_{x\neq y} \frac{|u(x)-u(y)|}{\|x-y\|^\beta}\right)^{\frac{1-\gamma}{1-\beta}} \cdot \left(\sup_{x\neq y} \frac{|u(x)-u(y)|}{\|x-y\|}\right)^{\frac{\gamma-\beta}{1-\beta}} \\
&\le \left(\|u\|_{L^\infty} + \sup_{x\neq y} \frac{|u(x)-u(y)|}{\|x-y\|^\beta}\right)^{\frac{1-\gamma}{1-\beta}} \cdot \left(\|u\|_{L^\infty} + \sup_{x\neq y} \frac{|u(x)-u(y)|}{\|x-y\|}\right)^{\frac{\gamma-\beta}{1-\beta}} \\
&= \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \cdot \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}
\end{aligned}$$

□

3.[Ex.3] Denote by U the open square $\{x \in \mathbb{R}^2 \mid |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0,\ |x_2| < x_1 \\ 1 + x_1 & \text{if } x_1 < 0,\ |x_2| < -x_1 \\ 1 - x_2 & \text{if } x_2 > 0,\ |x_1| < x_2 \\ 1 + x_2 & \text{if } x_2 < 0,\ |x_1| < -x_2 \end{cases}$$

For which $1 \le p \le \infty$ does u belongs to $W^{1,p}(U)$?

Proof. Obviously, $u \in L^p(U)$ for every $p \in [1, +\infty]$, we find the weak derivative of u.

Take $\varphi \in C_c^\infty(U)$, then we have

$$\begin{aligned}
&\int_U u(x) \cdot \partial_1\varphi(x)\ dx \\
&= \int_{-1}^1 \Big(\int_{|x_1|\ge|x_2|} (1-|x_1|)\cdot \partial_1\varphi\ dx_1 + \int_{|x_1|\le|x_2|} (1-|x_2|)\cdot\partial_1\varphi\ dx_1 \Big)\ dx_2 \\
&= \int_{-1}^1 \Big(\int_{|x_1|\ge|x_2|} (-|x_1|)\cdot \partial_1\varphi\ dx_1 + (-|x_2|)\int_{|x_1|\le|x_2|} \partial_1\varphi\ dx_1 \Big)\ dx_2 \\
&= \int_{-1}^1 \Big(|x_2|\cdot\varphi(|x_2|,x_2) + \int_{x_1>|x_2|} \varphi\ dx_1 - |x_2|\cdot\varphi(-|x_2|,x_2) - \int_{x_1<-|x_2|} \varphi\ dx_1 \\
&\quad + (-|x_2|)\cdot(\varphi(|x_2|,x_2) - \varphi(-|x_2|,x_2))\Big) dx_2 \\
&= \int_{-1}^1 \Big(\int_{x_1>|x_2|} \varphi\ dx_1 - \int_{x_1<-|x_2|} \varphi\ dx_1 \Big) dx_2 \\
&= \int_U (\mathbf{1}_{\{x_1>|x_2|\}} - \mathbf{1}_{\{x_1<-|x_2|\}})\varphi(x)\ dx
\end{aligned}$$

Hence, $-\partial_1 u = \mathbf{1}_{\{x_1>|x_2|\}} - \mathbf{1}_{\{x_1<-|x_2|\}}$ in weak sense. By the same discussion for $\partial_2 u$, we have

$$-\partial_1 u = \mathbf{1}_{\{x_1>|x_2|\}} - \mathbf{1}_{\{x_1<-|x_2|\}}$$
$$-\partial_2 u = \mathbf{1}_{\{x_2>|x_1|\}} - \mathbf{1}_{\{x_2<-|x_1|\}}$$

Thus $\nabla u \in L^p$ for every $p \in [1, +\infty]$. $u \in W^{1,p}(U)$ for every $p \in [1, +\infty]$ □

4.[Ex.4] Assume $n = 1$ and $u \in W^{1,p}(0, 1)$ for some $1 \le p < \infty$.

(a) Show that u is equal a.e. to an absolutely continuous function and u' (which exists a.e.) belongs to $L^p(0, 1)$.

(b) Prove that if $1 < p < \infty$, then

$$|u(x) - u(y)| \le |x - y|^{1-\frac{1}{p}} \Big(\int_0^1 |u'|^p dt \Big)^{1/p} \tag{4.1}$$

for a.e. $x, y \in [0, 1]$.

Proof. (a) Let g be the weak derivative of u. Consider $f(x) := \int_0^x g(t)\, dt$. Since $g \in L^p(0, 1)$, we have that f is absolutely continuous with $f' = g$ a.e. Now for $\forall \varphi \in C_c^\infty(0, 1)$, we have

$$\int_0^1 u \cdot \varphi' \, dx = -\int g \cdot \varphi = \int f \cdot \varphi' \tag{4.2}$$

which implies

$$\int_0^1 (u - f) \cdot \varphi' = 0 \implies u = f + C \ \text{ a.e. for some constant } C \tag{4.3}$$

Hence, u is absolutely continuous with $u' = g$ a.e.

(b) By conclusion from (a), we know $u(x) = \int_0^x u'(t)\, dt + C$. Then we have

$$\begin{aligned} |u(x) - u(y)| &\le \int_0^1 \mathbf{1}_{[x,y]} |u'(t)| \, dt \\ &\le \Big(\int_0^1 \mathbf{1}_{[x,y]} \Big)^{1-1/p} \Big(\int_0^1 |u'|^p \Big)^{1/p} = |x - y|^{1-\frac{1}{p}} \Big(\int_0^1 |u'|^p \Big)^{1/p} \end{aligned} \tag{4.4}$$

for a.e. $x, y \in [0, 1]$

□

5.[Ex.5] Let U, V be open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on V, $\zeta = 0$ near ∂U. (Hint: Take $V \subset\subset W \subset\subset U$ and mollify χ_W)

Proof. Since $V \subset\subset U$, $\bar{V}$ and ∂V are compact in U. For $\forall x \in \partial V$, there exists $r_x > 0$, such that $B(x, r_x) \subset U$. By compactness of ∂V, there exists finitely many $\{B(x_i, r_{x_i}/2)\}_{1\le i\le n}$, such that $\cup_{i=1}^n B(x_i, r_{x_i}/2) \supset \partial V$. Now define W to be

$$W = V \cup \Big(\bigcup_{i=1}^{n} B(x_i, r_{x_i}/2)\Big)$$

Then $V \subset\subset W \subset\subset U$. We choose r to be

$$r = \min\{\mathrm{dist}(\overline{V, W^c}), \mathrm{dist}(\overline{W}, U^c)\}/2$$

Let $\rho \in C_c^\infty(B(0, r))$ with $\int \rho = 1$, define $\zeta := \rho * \chi_W$. We show ζ satisfies. ζ is smooth since $\rho \in C_c^\infty$. For $x \in V$, $B(x, r) \subset W$, we have

$$\zeta(x) = \int_{B(x,r)} \chi_W(y)\rho(x-y)\ dy = 1$$

For $\mathrm{dist}(x, \partial U) < r/2$, $B(x, r) \bigcap \overline{W} = \varnothing$

$$\zeta(x) = \int_{B(x,r)} \chi_W(y)\rho(x-y)\ dy = 0$$

□

6.[Ex.6] Assume U is bounded and $U \subset\subset \bigcup_{i=1}^N V_i$. Show there exists C^∞ functions ζ_i $(i = 1, \ldots, N)$ such that

$$\begin{cases} 0 \le \zeta_i \le 1,\ \text{supp}\ \zeta_i \subset V_i \quad (i = 1, \ldots, N) \\ \displaystyle\sum_{i=1}^{N} \zeta_i = 1 \qquad \text{on } U \end{cases}$$

The functions $\{\zeta_i\}_{i=1}^N$ form a partition of unity.

Proof. Firstly, we prove that there exists $\{W_i\}_{1\le i\le N}$ such that $W_i \subset\subset V_i$, $U \subset\subset \bigcup_{i=1}^N W_i$. For $1 \le i \le N$. Define $\{W_{i,n}\}_{n\in\mathbb{N}}$ as

$$W_{i,n} := \{x \in V_i \mid\ \mathrm{dist}(x, \partial V_i) > \frac{1}{n}\}$$

Thus $V_i = \bigcup_{n\in\mathbb{N}} W_{i,n}$ and $\overline{U} \subset \bigcup_{1\le i\le N} V_i = \bigcup_{\substack{1\le i\le N\\ n\in\mathbb{N}}} W_{i,n}$. By compactness of $\overline{U}$, we could find finitely many W_{j,n_j} such that

$$\overline{U} \subset \bigcup_{j\ finite} W_{j,n_j}$$

Notice $W_{i,n} \subset W_{i,m}$ for $n < m$ for every i. By adding some W_{j,n_j} if necessary, we could deduce that

$$\overline{U} \subset \bigcup_{1\le i\le n} W_{i,n_i}$$

for some $n_i \in \mathbb{N}$. And $W_{i,n_i} \subset\subset V_i$.

Secondly, by conclusions from Ex.1, we could find $\tilde{\zeta}_i \in C_c^\infty(V_i)$ such that $\tilde{\zeta}_i \equiv 1$ in W_i, $\tilde{\zeta}_i \ge 0$. Notice also $\bigcup_{i=1}^N W_i \subset\subset \bigcup_{i=1}^N (\text{supp } \tilde{\zeta}_i)^\circ$. There exists $\eta \in C_c^\infty\big(\bigcup_{i=1}^N (\text{supp } \zeta_i)^\circ\big)$ such that $\eta \equiv 1$ in $\bigcup_{i=1}^N W_i$. And $0 \le \eta \le 1$. Then Let

$$\zeta_i = \eta \cdot \frac{\tilde{\zeta}_i}{\sum_{i=1}^N \tilde{\zeta}_i}$$

We could see ζ_i is well-defined and is smooth with support in V_i. And $\sum \zeta_i = 1$ on $\bigcup_{i=1}^N W_i$. Thus, $\{\zeta_i\}_{i=1}^N$ forms partition of unity in $\overline{U}$. □

7.[Ex.7] Assume that U is bounded and there exists a smooth vector field α such that $\alpha\cdot\nu \ge 1$ along ∂U, where ν as usual denotes the outward unit normal. Assume $1 \le p < \infty$.

Apply the Gauss-Green Theorem to $\int_{\partial U} |u|^p \alpha \cdot \nu \ dS$, to derive a new proof of the trace inequality

$$\int_{\partial U} |u|^p \ dS \le C \int_U |Du|^p + |u|^p \ dx \tag{7.1}$$

for all $u \in C^1(\bar{U})$.

Proof. Since $\alpha \cdot \nu \ge 1$, we have

$$\begin{aligned}\int_{\partial U} |u|^p \ dS \le \int_{\partial U} |u|^p \alpha \cdot \nu \ dS &= \int_U \text{div}(|u|^p \alpha) \ dx \\ &= \int_U p \ sgn(u) \ |u|^{p-1} Du \cdot \alpha + |u|^p \text{div}(\alpha) \ dx \\ &\le (p \cdot \sup|\alpha| + \sup|\text{div}(\alpha)|) \int_U |Du|^p + |u|^p\end{aligned}$$

where we apply Holder's inequality to the last inequality. Let $C = p \cdot \sup|\alpha| + \sup|\text{div}(\alpha)|$, we got (7.1) □

8.[Ex.8] Let U be bounded, with a C^1 boundary. Show that a "typical" function $u \in L^p(U)$ $(1 \leq p < \infty)$ does not have a trace on ∂U. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$.

Proof. We give out a counterexample. i.e. a sequence of $u_n \in C(\bar{U}) \cap L^p(U)$ such that $u_n \to u$ in $L^p(U)$ but $u_n|_{\partial U}$ does not converge to $u|_{\partial U}$ in $L^p(\partial U)$. This contradicts to continuity of T if such a T exists.
Let u_n be defined as:

$$u_n(x) := \begin{cases} 0, & \text{dist}(x, \partial U) \geq \dfrac{1}{n} \\ 1 - n \cdot \text{dist}(x, \partial U), & \text{dist}(x, \partial U) < \dfrac{1}{n} \end{cases} \tag{8.1}$$

Then $\|u\|_{L^p(U)} \leq m\{x \mid \text{dist}(x, \partial U) < 1/n\} \to 0$, where m denotes Lebesgue measure. $0 \in L^p(U) \cap C(\bar{U})$. However, $u_n|_{\partial U} \equiv 1$ does not converge to 0. □

9.[Ex.9] Integrate by parts to prove the interpolation inequality:

$$\|Du\|_{L^2} \leq C \cdot \|u\|_{L^2}^{1/2} \|D^2 u\|_{L^2}^{1/2}$$

for all $u \in C_c^\infty(U)$. Assume U is bounded, ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$.
(Hint: Take sequences $\{v_k\}_{k=1}^\infty \subset C_c^\infty(U)$ converging to u in $H_0^1(U)$ and $\{w_k\}_{k=1}^\infty \subset C^\infty(\overline{U})$ converging to u in $H^2(U)$.)

Proof. For $u \in C_c^\infty(U)$, by Holder's Inequality:

$$\int_U |Du|^2 \, dx = \int_U \nabla u \cdot \nabla u = -\int_U \Delta u \cdot u \leq C \cdot \|D^2 u\|_{L^2} \|u\|_{L^2}$$

Now for $u \in H_0^1 \cap H^2$, take sequences $v_k \in C_c^\infty(U)$ converging to u in H_0^1 norm, take sequences $w_k \in C^\infty(\overline{U})$ converging to u in H^2 norm. Then

$$\|v_k\|_{H^1} \to \|u\|_{H^1}, \quad \|w_k\|_{H^2} \to \|u\|_{H^2}$$

We claim that $Dv_k \cdot Dw_k \to |Du|^2$ in L^1 norm.

$$\int_U |Du|^2 - \int_U Dv_k \cdot Dw_k \leq \int_U |Du - Dv_k| \cdot |Dw_k| + \int_U |Du||Du - Dw_k| \to 0$$

By the same discussion, we have $v_k \cdot \Delta w_k \to u \cdot \Delta u$

$$\begin{aligned}\int_U |Du|^2 &= \lim_{k\to\infty} \int_U Dv_k \cdot Dw_k \\ &= \lim_{k\to\infty} \int_{\partial U} v_k Dw_k \cdot n \ ds - \int_U v_k \Delta w_k \\ &= -\int_U u \cdot \Delta u \leq C\|u\|_{L^2}\|D^2 u\|_{L^2}\end{aligned}$$

□

10.[Ex.10]

(a) Integrate by parts to prove

$$\|Du\|_{L^p} \leq C \cdot \|u\|_{L^p}^{1/2}\|D^2u\|_{L^p}^{1/2}$$

for $2 \leq p < \infty$ and all $u \in C_c^\infty(U)$.

(Hint: $\int_U |Du|^p \ dx = \sum_{i=1}^n \int_U u_{x_i} u_{x_i} |Du|^{p-2} \ dx$.)

(b) Prove

$$\|Du\|_{L^{2p}} \leq C \cdot \|u\|_{L^\infty}^{1/2}\|D^2u\|_{L^p}^{1/2}$$

for $1 \leq p < \infty$ and all $u \in C_c^\infty(U)$

Proof. (a) By integrate by parts and Holder's inequality:

$$\begin{aligned}\|Du\|_{L^p}^p &= \int_U \sum_{i=1}^n u_{x_i} u_{x_i} |Du|^{p-2} \ dx \\ &= \int_{\partial U} u \cdot |Du|^{p-2}\frac{\partial u}{\partial n} \ ds - \int_U u \ \nabla \cdot (|Du|^{p-2} Du) dx \\ &\leq C \cdot \int_U |u| \ |Du|^{p-2} \ |D^2u| dx \\ &\leq_{(Holder)} C \cdot \|u\|_{L^p} \cdot \|Du\|_{L^p}^{p-2} \cdot \|D^2u\|_{L^p}\end{aligned}$$

Thus $\|Du\|_{L^p}^2 \leq C \cdot \|u\|_{L^p} \cdot \|D^2\|_{L^p}$ for $2 \leq p < \infty$

(b) Similarly, by integrates by parts and Holder's Inequality, we have

$$\begin{aligned}\|Du\|_{L^{2p}}^{2p} &= \int_U |Du|^{2p} \ dx = \int_U \sum_i u_{x_i} u_{x_i} |Du|^{2p-2} \ dx \\ &= \int_{\partial U} u \ \sum_i u_{x_i} |Du|^{2p-2} \ ds - \int_U u \cdot \nabla \cdot (|Du|^{2p-2} Du) \ dx \\ &\leq C \cdot \int_U |u| \cdot |Du|^{2p-2} \cdot |D^2 u|\end{aligned}$$

$$\leq C \cdot \|u\|_{L^\infty} \cdot \|Du\|_{L^{2p}}^{2p-2} \cdot \|D^2 u\|_{L^p}$$

□

11.[Ex.11] Suppose U is connnected and $u \in W^{1,p}(U)$ satisfies:

$$Du = 0 \quad a.e. \text{ in } U$$

Prove u is a constant a.e. in U.

Proof. Take mollifiers η_ϵ with support in $B(0,\epsilon)$. $U_\epsilon = \{x \in U \mid \text{dist }(x, \partial U) > \epsilon\}$. We have $u * \eta_\epsilon \in C^\infty(U_\epsilon)$. $D(u * \eta_\epsilon) = (Du) * \eta_\epsilon = 0$ a.e. Thus, $u * \eta_\epsilon = C(\epsilon)$ is a constant.
Then $u * \eta_\epsilon \to u$ a.e. in U. Since $u * \eta_\epsilon$ is a sequence of constants, they must converge to a constant. Let $C(\epsilon) \to C$ as $\epsilon \to 0$. Then $u = C$ a.e. □

12.[Ex.12] Show by example that if we have $\|D^h u\|_{L^1(V)} \leq C$ for all $0 < |h| < \frac{1}{2}\text{dist}(V, \partial U)$, it does not necessarily follow that $u \in W^{1,1}(V)$.

Proof. Let $U = (-2, 2) \subset \mathbb{R}$, $V = (-1, 1)$. Define u as

$$u = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} \implies D^h(u) = \begin{cases} 1/h & x \in (0,h) \\ 0 & x \notin (0,h) \end{cases}$$

Thus $\|D^h\|_{L^1(V)} = 1$ for $|h| < \frac{1}{2}\text{dist}(V, \partial U)$. But $u \notin W^{1,1}(V)$. □

13.[Ex.13] Give an example of an open set $U \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(U)$, such taht u is not Lipschitz continuous in U. (Hint: Take U to be the open unit disk in $\mathbb{R}^2$, with a slit removed).

Proof. Consider $n = 2$, let $U = B_1(0) \backslash (\{x \leq 0, y = 0\} \bigcup B_{1/2}(0))$ be an annulus with a slit removed. Let u be defined by polar coordinates:

$$u(r, \theta) = r \cdot \theta, \qquad \theta \in (0, 2\pi)$$

The Jacobian Matrix between (x, y) and (r, θ) is

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r \in (\frac{1}{2}, 1)$$

Then obviously $u \in L^1(U)$ with derivatives $\partial_r u(r,\theta) = \theta,\ \partial_\theta u(r,\theta) = r$, then $(\partial_x u, \partial_y u) = J^{-1}(\partial_r u, \partial_\theta u)$ bounded. But for $(r_1,\theta_1) = (\frac{3}{4}, 2\pi - \varepsilon)$, $(r_2,\theta_2) = (\frac{3}{4}, \varepsilon)$, we have

$$\frac{|u(r_1,\theta_1) - u(r_2,\theta_2)|}{\|(r_1,\theta_1) - (r_2,\theta_2)\|} \to \infty \qquad \varepsilon \to 0$$

Thus, u is not Lipschitz continuous in U. □

14.[Ex.14] Verify taht if $n > 1$, the unbounded function $u = \log\log(1 + \frac{1}{|x|}$ belongs to $W^{1,n}(U)$, for $U = B^0(0,1)$.

Proof. Firstly, $u \in L^n$:

$$\int_U |u|^n\ dx = \alpha(n) \int_0^1 |\log\log(1 + \frac{1}{r})|^n \cdot r^{n-1}\ dr < \infty$$

Where $\alpha(n)$ denotes the volume of $B_1(0) \subset \mathbb{R}^n$.
Secondly, $\partial u \in L^n$, take $\partial_{x_1} u$ for example:

$$\begin{aligned}\int_U |\partial_{x_1} u|^n\ dx &= \int_U |\frac{1}{\log(1+\frac{1}{r}) \cdot (1 + \frac{1}{r})} \cdot (-\frac{x_1}{r^3})|^n\ dx \\ &\le \alpha(n) \int_0^1 |\frac{1}{(r^2 + r) \cdot \log(1 + \frac{1}{r})}|^n \cdot r^{n-1}\ dr \\ &\le C \int_0^{\frac{1}{2}} \frac{1}{r \cdot |\log(\frac{1}{r})|^n}\ dr < \infty\end{aligned}$$

Thus $u \in W^{1,n}(U)$ □

15.[Ex.15] Fix $\alpha > 0$ and let $U = B^0(0,1)$. Show there exists a constant C, depending only on n and α, such that

$$\int_U u^2\ dx \le C \int_U |Du|^2\ dx \tag{15.1}$$

Provided

$$|\{x \in U \mid u(x) = 0\}| \ge \alpha\ , \quad u \in H^1(U). \tag{15.2}$$

Proof. We prove by contradiction, assume there exists a sequence of $u_k \in W^{1,p}(U)$, satisfies (15.2), and has the property of

$$\int_U u_k^2\ dx > k \int_U |Du_k|^2$$

Then define

$$v_k := \frac{u_k}{\|u_k\|_2}, \qquad \text{then}\ \ \|v_k\|_2 = 1,\ \|Dv_k\|_2 < \frac{1}{k}.$$

Since v_k are bounded in $H^1(U)$, there exists a subsequence of v_{k_j} and $v \in H^1$ such that

$$v_{k_j} \to v \text{ in } L^2, \quad v_k \rightharpoonup v \text{ in } W^{1,2}$$

Then for $\forall\ i = 1, \dots, n$. $\varphi \in C_c^\infty(U)$,

$$|\int_U v \cdot \partial_i \varphi| = |\lim_{j\to\infty} \int_U -\partial_i v_{k_j} \cdot \varphi| \le \|\varphi\|_2 \cdot \|Dv_{k_j}\|_2 \to 0$$

Thus $Dv \equiv 0$ a.e. Thus, v is a constant function, and $\|v\|_2 = \lim_{k\to\infty} \|v_k\|_2 = 1$, thus $v \equiv C_0 \neq 0$ a.e. But

$$\int_U |v - v_{k_j}|^2 \ge C_0^2\ |\{x \in U \mid v_{k_j} = 0\}| \ge C_0^2 \alpha > 0$$

That's contradiction to $v_{k_j} \to v$ in $L^2(U)$. □

16.[Ex.16] (Variant of Hardy's inequality) Show that for each $n \ge 3$ there exists a constant C so that

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \le C \int_{\mathbb{R}^\kappa} |Du|^2 dx \tag{16.1}$$

for all $u \in H^1(\mathbb{R}^n)$. (Hint: $|Du + \lambda \frac{x}{|x|^2} u|^2| \ge 0$ for each $\lambda \in \mathbb{R}$.

Proof. Apply Hardy's inequality [See page 296]: if $v \in H^1(B(r))$, then we have

$$\int_{B(r)} \frac{v^2}{|x|^2} dx \le C \int_{B(r)} |Dv|^2 + \frac{v^2}{r^2}\ dx \tag{16.2}$$

Now Let u_n be defined on $B(n)$ by $u_n = u|_{B(n)}$, then we have

$$\int_{B(n)} \frac{u^2}{|x|^2}\ dx \le C \int_{B(r)} |Du|^2 + \frac{u^2}{r^2}\ dx \le C \int_{\mathbb{R}^n} |Du|^2\ dx + \frac{\|u\|_2}{n^2}$$

Let $n \to \infty$, we got the Hardy's Inequality

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\ dx \le C \int_{\mathbb{R}^n} |Du|^2\ dx$$

□

17.[Ex.17] (Chain rule) Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose U is bounded and $u \in W^{1,p}(U)$ for some $1 \le p \le \infty$. Show

$$v := F(u) \in W^{1,p}(U) \quad \text{and} \quad v_{x_i} = F'(u) u_{x_i} \quad (i = 1, \dots, n). \tag{17.1}$$

Proof. By density of $W^{1,p}(U)$, choose $u_n \in C^\infty(\overline{U})$ such that $u_n \to u$ in $W^{1,p}(U)$. Now let $v_n := F(u_n)$, then $\partial_i v_n = F'(u_n) \cdot \partial_i u_n$. Notice F is C^1 with F' bounded, thus $F(u_n) - F(u) \le \|F'\|_\infty \cdot |u_n - u|$. Hence,

$$\|v_n - v\|_p \le \|F'\|_\infty \cdot \|u_n - u\|_p \to 0.$$
$$Dv_n \to F'(u)Du \text{ a.e.} \quad \|Dv_n\|_p \le \|F'\|_\infty \cdot \|Du_n\|$$

Since Du_n converges to Du in L^p norm, thus Du_n are bounded in L^p norm. By Dominated Convergence Theorem, we have $Dv_n \to F'(u)Du$ in $L^p(U)$. Thus $v_n \to v$ in $W^{1,p}(U)$, with $Dv = F'(u)Du$. □

18.[Ex.18] Assume $1 \le p \le \infty$ and U is bounded.

(a) Prove that if $u \in W^{1,p}(U)$, then $|u| \in W^{1,p}(U)$

(b) Prove $u \in W^{1,p}(U)$ implies $u^+, u^- \in W^{1,p}(U)$, and

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \le 0\} \end{cases} \tag{18.1}$$

$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \ge 0\} \\ -Du & \text{a.e. on } \{u < 0\} \end{cases} \tag{18.2}$$

(Hint: $u^+ = \lim_{\varepsilon\to 0} F_\varepsilon(u)$ for

$$F_\varepsilon(z) := \begin{cases} (z^2+\varepsilon^2)^{1/2} - \varepsilon & \text{if } z \ge 0 \\ 0 & \text{if } z < 0 \end{cases} \tag{18.3}$$

(c) Prove that if $u \in W^{1,p}(U)$, then

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$

Proof. For (a) and (b):
Consider F_ε defined as (18.3). Then $u^+ = \lim_{\varepsilon\to 0} F_\varepsilon(u)$. Notice $|u| = 2u^+ - u$, to prove $|u| \in W^{1,p}$, we just need to prove $u^+ \in W^{1,p}$. Firstly we observe

$$F_\varepsilon'(z) = \begin{cases} \dfrac{z}{(z^2+\varepsilon^2)^{1/2}} & \text{if } z \ge 0 \\ 0 & \text{if } z < 0 \end{cases}, \qquad |F_\varepsilon'| \le 1, \quad F_\varepsilon(z) \overset{\varepsilon\to 0}{\to} \mathbf{1}_{z\ge 0}$$

Thus by conclusion from last problem, we have $F_\varepsilon(u) \in W^{1,p}$ with $\partial_i(F_\varepsilon(u)) = F_\varepsilon'(u)\partial_i u \to \mathbf{1}_{u>0}\partial_i u$ as $\varepsilon \to 0$.

Since $|F'_\varepsilon|$ is uniformly bounded by dominated convergence theorem, we have

$$\partial_i(F_\varepsilon(u)) \to \mathbf{1}_{u>0}\partial_i u \quad \text{in } L^p(U)$$

$F_\varepsilon(u) \le |u|$ is bounded, by DCT again we have $u^+ = \lim_{\varepsilon\to 0} F_\varepsilon(u)$ in $L^p(U)$. Thus, $F_\varepsilon(u) \to u^+$ in $W^{1,p}$ with $Du^+ = \mathbf{1}_{u>0}Du$. Du^- is given similarly.

For (c):

Notice $u = u^+ - u^-$ with $u^+, u^- \in W^{1,p}$. $Du^+ = Du^- = 0$ a.e. on the set $\{u = 0\}$. Thus $Du = Du^+ - Du^- = 0$ a.e. on the set $\{u = 0\}$ □

19.[Ex.19] Provide details for the following alternative proof that if $u \in H^1(U)$, then

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$

Let ϕ be a smooth, bounded, and nondecreasing function, such that ϕ' is bounded and $\phi(z) = z$ if $|z| \le 1$. Set

$$u^\varepsilon := \varepsilon\phi(u/\varepsilon). \tag{19.1}$$

Show that $u^\varepsilon \rightharpoonup 0$ weakly in $H^1(U)$ and therefore

$$\int_U Du^\varepsilon \cdot Du\ dx = \int_U \phi'(u/\varepsilon)|Du|^2\ dx \to 0$$

Employ this observation to finish the proof.

Proof. u^ε is defined as (19.1), then $u^\varepsilon \le \varepsilon \cdot \|\phi\|_\infty \to 0$. Thus, $u^\varepsilon \to 0$ in $L^2(U)$. And $\phi(0) = 0$, ϕ is non-decreasing shows $\phi' \ge 0$. u^ε satisfies

$$\{u^\epsilon = 0\} = \{u = 0\}, \qquad D(u^\varepsilon) = \phi'(u/\varepsilon)\ Du$$

Since ϕ' is bounded, $\|Du^\epsilon\|_2 \le \|\phi'\|_\infty \cdot \|Du\|_2$ are bounded. Thus there exists a sequence of ε_n, such that $u^{\varepsilon_n} \rightharpoonup v$ weakly in $H^1(U)$ and $u^{\epsilon_n} \to v$ in $L^2(U)$ for some $v \in H^1(U)$. Notice that $u^\varepsilon \to 0$ in $L^2(U)$, thus $v = 0$ in U. Therefore,

$$0 = \lim_{n\to\infty}\int_U Du^{\epsilon_n} \cdot Du = \lim_{n\to\infty}\int_U \phi'(u/\varepsilon_n)|Du|^2 \ge \int_{\{u=0\}} |Du|^2 \ge 0$$

Thus $|Du| = 0$ a.e. in $\{u = 0\}$ □

20.[Ex.20] Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$, with the bound

$$\|u\|_{L^\infty} \le C\|u\|_{H^s} \tag{20.1}$$

for a constant C depending only on s and n.

Proof. $u \in H^s(\mathbb{R}^n)$, we have

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} (1+|y|^s)^2 |\hat{u}(y)|^2 \, dy < \infty \tag{20.2}$$

Then

$$\begin{aligned} |u(x)| &= \frac{1}{(2\pi)^{\frac{n}{2}}} \Big| \int_{\mathbb{R}^n} e^{ixy} \hat{u}(y) dy \Big| \\ &\le \frac{1}{(2\pi)^{n/2}} \Big(\int_{\mathbb{R}^n} \frac{1}{(1+|y|^s)^2} \, dy \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} (1+|y|^s)^2 |\hat{u}(y)|^2 dy \Big)^{\frac{1}{2}} \\ &\le \frac{\|u\|_{H^s}}{(2\pi)^{n/2}} \Big(|B_1| + \int_{|y|>1} \frac{1}{(1+|y|^{s/2})^2} \, dy \Big) = C\|u\|_{H^s} \end{aligned}$$

where B_1 is the unit ball of $\mathbb{R}^n$, $C < \infty$ Since

$$\int_{|y|>1} \frac{1}{(1+|y|^s)^2} \, dy \le \int_{|y|>1} \frac{1}{|y|^{2s}} \, dy = |\partial B_1| \int_{r>1} \frac{1}{r^{2s-n+1}} \, dr = \frac{|\partial B_1|}{2s-n} < \infty \tag{20.3}$$

for every $x \in \mathbb{R}^n$. Thus $u \in L^\infty$ with $\|u\|_{L^\infty} \le C\|u\|_{H^s}$, where C is bounded by

$$C \le \frac{1}{(2\pi)^{n/2}} (|B_1| + \frac{|\partial B_1|}{2s-n}) \tag{20.4}$$

depending only on s and n. □

21.[Ex.21] Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $uv \in H^s(\mathbb{R}^n)$ and

$$\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s} \tag{21.1}$$

the constant C depending only on s and n.

Proof. Notice $\widehat{uv} = \frac{1}{(2\pi)^{n/2}} \hat{u} * \hat{v}$, and there exists C depending only on n and s such that

$$(1+|y|^s) \le C \cdot [(1+|y-z|^s) + (1+|z|^s)] \quad \text{for } \forall z \in \mathbb{R}^n \tag{21.2}$$

Hence, we have

$$\begin{aligned}
\|uv\|_{H^s}^2 &= \int (1+|y|^s)^2 |\widehat{uv}(y)|^2 \ dy = \frac{1}{(2\pi)^n} \int (1+|y|^s)^2 |\hat{u} * \hat{v}(y)|^2 \ dy \\
&= \frac{1}{(2\pi)^n} \int (1+|y|^s)^2 \Big| \int \hat{u}(y-z)\hat{v}(z) \ dz \Big|^2 \ dy \\
&= \frac{1}{(2\pi)^n} \int \Big| \int (1+|y|^s) \hat{u}(y-z)\hat{v}(z) \ dz \Big|^2 \ dy \\
&\leq C \int \Big| \int (1+|y-z|^s)\hat{u}(y-z)\hat{v}(z) + (1+|z|^s)\hat{u}(y-z)\hat{v}(z) \ dz \Big|^2 \ dy \\
&= C \int (|f\hat{u}| * \hat{v} + \hat{u} * |f\hat{v}|)^2 \ dx \quad \text{, where } f(y) = (1+|y|^s) \\
&\leq_{(Young)} C \cdot (\|u\|_{H^s}^2 \|\hat{v}\|_{L^1}^2 + \|v\|_{H^s}^2 \|\hat{u}\|_{L^1}^2)
\end{aligned}$$

Notice that

$$\|\hat{u}\|_{L^1}^2 = (\int |\hat{u}(x)| \ dx)^2 \leq \Big(\int \frac{1}{(1+|x|^s)^2} \ dx \Big) \Big(\int (1+|x|^s)^2 |\hat{u}^2| \ dx \Big) \leq C\|u\|_{H^s} \tag{21.3}$$

Thus, we conclude that

$$\|uv\|_{H^s} \leq C\|u\|_{H^s}\|v\|_{H^s}$$

□